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THE NATURE and LAWS OF CHANCE.



Containing, among other Particulars,

- THE Solutions of several abstruse and important Problems. ✧
- THE Doctrine of Combinations and Permutations clearly deduced. ✧
- A NEW and comprehensive Problem of great Use in discovering the Advantage or Loss in Lotteries, Raffles, &c. ✧
- A CURIOUS and extensive Problem on the Duration of Play. ✧
- PROBLEMS for determining the Probability of winning at Bowls, Coits, Cards, &c. ✧
- A PROBLEM for finding the Trials wherein it may be undertaken ✧
- that a proposed Event shall happen or fail a given Number of Times. ✧
- A PROBLEM to find the Chance for a given Number of Points on a given Number of Dice. ✧
- FULL and clear Investigations of two Problems, added at the End of Mr *De Moivre's* last Edition; one of them allowed by that great Man to be the most useful on the Subject, but their Demonstrations there omitted. ✧
- Two new Methods for summing of Series. ✧

THE WHOLE

After a new, general, and conspicuous Manner,

And illustrated with

A great VARIETY of EXAMPLES.

By **THOMAS SIMPSON,**

Teacher of the Mathematicks,

Printed by EDWARD CAVE, at *St John's Gate*. 1749.

And sold by the Booksellers.

Emory H. B. 7/6

OWAL LTD KATULAT

TO

EDITH

1910

Dear Edith,

I have just received your letter of the 10th inst.

and am glad to hear from you.

I am well and hope this finds you the same.

I have not much news to write at present.

I am sure you will be interested to hear

that I am still in the same old place.

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P R E F A C E.



As it is reasonable to expect that one, who undertakes to lay his Thoughts before the Publick on any Subject, should be able to point out some Advantage to be reaped from it, it may not be improper, first to shew upon what Ground I was induced to hope that the few following Sheets might be of Service, and claim a Reception. And here, I must ingenuously own that what had the greatest Weight, was not any immoderate Opinion of my own Productions, but the Scarcity and high Price of some, and the Imperfection of the rest of the few Books at present among us on the Subject; most of these being wholly taken up with a few low Problems, and, what is still a greater Defect, without affording any General Principles whereby the Thing might be extended farther. There is indeed but one, that I have met with, entirely free from this Objection; and, tho' it neither wants Matter nor Elegance to recommend it, yet the Price must, I am sensible, have put it out of the Power of many to purchase it; and even some, who want no Means to gratify their Desires this way, and who might not be inclinable to subscribe a Guinea for a single Book, however excellent, may not scruple the bestowing of a small

Matter on one, that perhaps may serve equally well for their Purpose.

I am satisfied, it may be deem'd a sort of Presumption to attempt, upon any Account, a Subject like this, after so great a Man as Mr De Moivre; and that some, who would pass for considerable Judges, have not been wanting to censure and condemn this Performance already on that Head, before having the least Knowledge of the Particulars therein contained. But they would do well to reflect, whether to proceed thus, may not prove a greater Impeachment of their own Judgment and Conduct, even among their Friends, than of his, whose Work, without any Offence given, they endeavour to depreciate.

I have perhaps as high an Opinion of that learned Author's Productions as any of those Gentlemen. But does it follow, because one Person has done well on this Subject, that nothing farther can be necessary? Besides, it is not every one that has a Genius fitted for the most exalted Speculations, or that is capable of reading the Works of the most sublime and celebrated Authors; and, therefore, tho' I should go no farther than to bring down some of the best and most useful Things already known to the Level of ordinary Capacities, I should think this might, in some measure, exempt me from Censure. However, I would not have the Reader conclude from hence, that he has nothing more to expect in these few Sheets than a bare Collection of low Matters; or that, like some of our present Mathematical Writers, I should be poorly ambitious of appearing the Author of a Performance, that would, was every Bird to claim his own Feather, be stript as naked as the Jay in the Fable.

It

It is not my Intention to say a great deal in behalf of this Work, knowing that, if there be a real Merit, it will best recommend itself, and that, without this, it will be in vain for me to hope for a Reception: Nevertheless, it may not be amiss to point out some of the most useful and curious Propositions therein, and make such useful Observations thereon, as may be of some Service in, or shew it not unworthy, the Perusal.

The 1st is a Proposition of Use thro'out the Work, and therefore ought to be well understood by a young Beginner; towards which the six succeeding Cases, in Illustration thereof, will not a little contribute, since, by them alone, a Person, but tolerably skill'd in common Arithmetick, may soon arrive to some Proficiency in the Subject. In the 2d and 3d Problems, besides other Things, the Doctrine of Combinations and Permutations is clearly and fully deduced. Nor is the 5th, shewing the Probability of an assigned Event happening a given number of Times in a given number of Tryals, for general Use, inferior to either of them. The 6th is very comprehensive, and of great Use in Lotteries, Cards, &c. And the 15th, for finding the Tryals wherein one may undertake that a proposed Event shall happen a given Number of Times, has been long look'd on as a Problem of the greatest Note and Consequence, and is solv'd in a more general Manner than hitherto. The 16th, 17th, and 20th, are also Problems of some Note and Difficulty. But the 22d to find the Chances for a given Number of Points on a given Number of Dice, and the 25th on the Duration of Play, are two of the most intricate and remarkable in the Subject, and both solv'd by Methods entirely new. The 27th is a Problem that was proposed to the Public some time ago in Latin, as a very difficult one, and has not (that I know of) been answered before. And the 24th and 30th are.

are the same with the two new ones, added in the End of Mr De Moivre's last Edition, whose Demonstrations that learned Author was pleased to reserve to himself, and are here fully and clearly investigated; which Problems are both of considerable Importance. Lastly, in the Lemmas, among other Things, will be found two new Methods for summing of Series; and by Help of one of them the Value of a Series of Powers, whether whole or broken, is determined in a more concise and general Manner than heretofore.

E R R A T A.

P. Age 2. l. 20. for $\frac{1}{2}$ 20. r. $\frac{1}{2}$ of 20. P. 13. l. 1. for $\frac{a}{a+b}$ r. $\frac{b}{a+b}$. P. 13. l. 11. for will r. &c. will. P. 17. l. 19. for Combination r. Continuation. P. 18. l. 14. for first r. first of the. P. 31. l. 18. for or r. &c. or. P. 36. l. 2. for the 2. n. the Sum of the 2. P. 36. l. 13. for true Value r. Value. P. 42. l. 21. for Places r. Place. P. 43. l. 1. for betaken r. be taken. P. 53. l. 15. for $q \times \frac{q-1}{2}$, &c. r. $-q \times \frac{q-1}{2}$, &c. P. 54. l. 4. for the Points r. the Number of Points. P. 56. l. 2. for Probability of r. Chances for. P. 70. l. 2. for $\frac{A}{300}$ r. $\frac{A}{360}$. P. 79. Line last, for Lem. 4. r. Lem. 5. P. 80. l. 5. for $=$, 2. r. $=$ 2.

N. B. By an Alteration inadvertently made in the 7th Problem, at the Correction of the Press, without remembering that the Sense of the two succeeding ones depended thereon, it may seem at first sight, as if the Solutions to those two Problems would hold nearly true only in very great Numbers; whereas they are in all Cases exactly so, and were so intended to be understood; as will appear from the Perusal.



O F
C H A N C E S.



DEFINITION I.



THE *Probability* of the Happening of an Event is to be understood as the Ratio of the Chances by which that Event may happen, to all the Chances by which it may either happen or fail.

As, supposing it were required to express the Probability of throwing either an Ace or Duce at the first Throw with a single Die; then there being in all 6 different Chances or Ways that the Die may fall, and only 2 of them for the Ace or Duce to come upward, the Probability of the Happening of one of these will be $\frac{2}{6}$ or $\frac{1}{3}$, and that of the contrary $\frac{4}{6}$ or $\frac{2}{3}$: Or, more generally, supposing there be a Chances for the Happening of an Event, and b Chances for the contrary; then

The NATURE and

the Probability of Happening will be $\frac{a}{a+b}$, and that of

Failing $\frac{b}{a+b} = 1 - \frac{a}{a+b}$.

C O R O L L A R Y.

HENCE it appears, that if the Probability of the Happening of an Event be subtracted from Unity, the Remainder will be the Probability of its Failing, and *vice versa*.

D E F I N I T I O N II.

THE *Expectation* on an Event is consider'd as the *present*, certain Value, or Worth of whatever Sum or Thing is depending on the Happening of that Event, and is compounded of that Sum and the Probability of obtaining it.

C O R O L L A R Y.

THEREFORE, if the Expectation on an Event be divided by the Value of the Thing expected on the Happening of that Event, the Quotient will be the Probability of Happening.

E X A M P L E.

SUPPOSE *A* to throw once with a single Die, on Condition that if either an Ace or Duce comes up he shall be intitled to 20 Shillings; then, because the Probability of his receiving the said Sum is $\frac{2}{6}$ (*Def. I.*) $\frac{2}{6} \times 20$ s. or $\frac{20}{3}$ s. will be the Expectation in this Case.

P R O B L E M I.

TO find the Probability that two assigned Events shall both happen.

Let

Let the Probability of the Happening of the first of the two Events be denoted by $\frac{a}{a+b}$, and that of the second by $\frac{c}{c+d}$; and suppose the Happening of both to entitle a Person *B* to the Sum \mathcal{S} . Now if the first of these should happen, it is manifest that, from that Time till the second is determined, the Expectation of *B* will be $\frac{c}{c+d} \times \mathcal{S}$, or so much is the Sum that he might in that Circumstance receive as an Equivalent for his Chance of obtaining the Sum \mathcal{S} . But the Probability of getting into this Circumstance, or being intitled to the Value $\frac{c}{c+d} \times \mathcal{S}$, being only $\frac{a}{a+b}$, his Expectation therefore, before either of the Events is decided, can be only $\frac{a}{a+b}$, part, of $\frac{c}{c+d} \times \mathcal{S}$, or $\frac{a}{a+b} \times \frac{c}{c+d} \times \mathcal{S}$, and therefore the required Probability of receiving it, or the Happening of both the Events, only $\frac{a}{a+b} \times \frac{c}{c+d}$; that is, the Probability that any two assigned Events shall both happen, will be equal to the Product of the Probabilities of the Happening of those Events considered separately.

C O R O L L A R Y.

WHEREFORE, since the Probability of the Happening of each of these Events may be compounded of the Probabilities of the Happening of two others, as well as that of receiving the Sum \mathcal{S} is of them two, &c. it follows that the Probability of the Happening of any given Number of Events, *i. e.* that they shall all happen, is equal to the Product

duct of all the Probabilities of Happening of those Events considered singly.

But as this Conclusion is the Basis whereon all the succeeding Calculations are founded, it may not be improper to enlarge a little farther thereon, and endeavour to render the same still more familiar and easy by a Numerical Exegesis, in order to shew those who are not so well acquainted with Algebraic Computation how to reason with Certainty on the Subject.

I. Suppose *A*, holding a single Die, to begin to throw, on Condition that if an Ace comes up both the first and second Throws he shall receive 1*l*. Now if an Ace should come up the first Throw, the Expectation, or Worth of his Chance would, it is manifest, then be $\frac{1}{6}$ of 1*l*; but the Probability of getting into that Circumstance being but $\frac{1}{6}$ (*Def. I.*) the required Expectation before he begins to throw will be only $\frac{1}{6}$ of $\frac{1}{6}$ of 1*l*. This divided by 1*l*. gives $\frac{1}{6} \times \frac{1}{6}$ for the Probability of receiving the Sum proposed, (*Cor. to Def. II.*) equal to the Product of the Probabilities of Happening of two Aces, when considered separately.

II. *B* upon certain Considerations agrees to deposit to *A* the Sum of 50 Shillings, if in the three first Throws with a single Die the latter throws three Aces. If an Ace should come up the first Throw, it is evident, from the last Case, that the Expectation of *A* would be $\frac{1}{6}$ of $\frac{1}{6}$ of 50 Shillings; but as the Probability of that Happening is only $\frac{1}{6}$, his Expectation before the first Throw will therefore be but $\frac{1}{6}$ of $\frac{1}{6}$ of $\frac{1}{6}$ of 50 Shillings; and consequently the Probability of his receiving the proposed Sum $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} (= \frac{1}{216})$ equal to the Product of the Probabilities of the Happening of the three component

nent Events considered separately; agreeable to the general Corollary.

Note, If, instead of throwing an Ace three times together with a single Die, the Condition had been to throw three Aces at once with three Dice, the Expectation would have been the very same: And it may be farther observed, that in divers other Cases, where one Event is, or may be conceived to be, compounded of several others more simple, which are all decided at the same Time, as in throwing of Dice, &c. it will facilitate the Reasoning very much, to suppose them determined one by one in a successive Order.

III. Imagine a Heap of 16 Counters, whereof 6 are red, and the rest black; and a Person to draw out 2 of them blindfold: To find the Odds that one or both of those shall be red ones. Let them be supposed to be taken one at a time, and that, if either of them prove red ones, the Drawer shall be entitled to a given Sum, as 1*l*. If the first drawn should happen to be red, his Expectation on the second would vanish; for as the Happening of one red one insures to him the proposed Sum, it cannot be of the least Advantage to draw another red one afterwards; and therefore the whole Expectation on the second Counter is compounded of, or depends on, the Probability of drawing a black one first, and a red one next after. Now the Probability of drawing a black one first, is $\frac{10}{16}$; and if a black one should be drawn, there being then only 15 Counters remaining, the Probability of taking a red one next, it is manifest, would be $\frac{6}{15}$, and the Expectation thereon $\frac{6}{15}$ of 1*l*. But the Probability of getting into that Circumstance being only $\frac{10}{16}$, the true Expectation on the second Counter is therefore only $\frac{10}{16}$ of $\frac{6}{15}$ of 1*l*. which added to $\frac{6}{16}$ of 1*l*. his Expectation on

the first, gives $\frac{1}{3}$ of 1. for the total ; wherefore the Probability that one or both the Counters shall be red ones is $\frac{2}{3}$, and the required Odds as 5 to 3. But this may be determined with more Facility, by first finding the Probability that neither of the two Counters shall be red ones ; for if the first should come out a black one, of which the Probability is $\frac{1}{10}$, then there remaining but 15 Counters, the Probability of taking a black one next would be $\frac{2}{15}$; which drawn into $(\frac{10}{16})$ the foregoing (*See the Cor.*) gives $\frac{1}{8}$ for the Probability that both the first and second shall be black ones ; and therefore the Odds are 5 to 3, as before found.

IV. Let there be a Lottery, consisting of 100 Tickets, wherein there are four Prizes : To find the Probability that in the three first Numbers that are drawn there shall be one or more Prizes. Here, as in the last Case, it will be convenient to find first the Probability that none of those Numbers shall be Prizes : In order to which, let the first be supposed to have come out a Blank ; then there being 99 Tickets remaining, and 95 of them Blanks, the Probability of drawing a Blank next will, it is manifest, be $\frac{95}{99}$; which therefore multiplied by $\frac{96}{100}$, the Probability of the first coming out as supposed, yields $\frac{96}{100} \times \frac{95}{99}$ for the Probability that the two first Tickets that are drawn shall be both Blanks. If both these should happen to be so, the Probability of taking a Blank next will be $\frac{94}{98}$, which therefore drawn into $\frac{96}{100} \times \frac{95}{99}$ gives $(\frac{96}{100} \times \frac{95}{99} \times \frac{94}{98} = \frac{7144}{8085})$ the Probability of drawing all three Blanks ; this subtracted from Unity (*Cor. to Def. I.*) leaves $\frac{941}{8085}$ for the required Probability of taking one or more Prizes ; wherefore the Odds that all the three are Blanks, are as 7144 to 941, or as 15 to 2 nearly.

V. *B* holding four common Dice, bets one Guinea to two, that

that one Ace, and no more, comes up the first Throw: To find his Advantage or Disadvantage. Let the Dice be conceived to be thrown one by one: Then if the first should come up an Ace, the Probability of missing the Ace all the next three Throws being $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$ (*Case II.*) the Probability that the first comes up an Ace, and all the rest otherwise, is $\frac{1}{6}$ of $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{125}{1296}$. But since this is only one of four Ways by which *B* may win, or because there is the same Chance that the 2d, or 3d, or 4th, may come up an Ace, and the rest otherwise, $\frac{125}{1296} \times 4 = \frac{500}{324}$ will, it is evident, be the Probability of *B*'s winning; wherefore his Expectation on the three Guineas, is $\frac{500}{324}$ of that Sum, from which deducting his own Stake, there remains $\frac{17}{108}$ of a Guinea = 3s. 3d. $\frac{2}{3}$ for the required Advantage, or so much is the Sum that he might give, upon an Equality of Chance, to another Person to lay him the same Wager.

VI. One with a single Die proposes to throw the Ace twice before either the Duce or Tray comes up once; To find the Odds against him. There is given the Probability that the Ace comes up the first significant Throw, where either Ace, Duce or Tray must come up, equal to $\frac{1}{3}$, which is also the Probability of the same Thing happening the second significant Throw; wherefore the Probability that an Ace shall come up both those Throws, it is evident, will be $\frac{1}{3}$ of $\frac{1}{3}$, or $\frac{1}{9}$, and the required Odds as 8 to 1.

P R O B L E M II.

LET any given Number (*n*) of Letters, as *A, B, C, D, E,* &c. or Things represented by them, be disposed in a regular Order; and let a given Number (*p*) of them be taken, one by one, as it happens: To find the Probability that they shall

shall come out according to the very Order in which they are placed; as *A* 1st, *B* 2d, *C* 3d, &c.

SOLUTION.

SINCE there are n Letters in all, the Probability of drawing *A* first is $\frac{1}{n}$. If *A* should be so drawn, then there remaining $n-1$ Letters, the Probability of taking *B* next will be $\frac{1}{n-1}$, whence (*Cor. to Prob. I.*) it is manifest that the Probability that *A* is taken first and *B* next, is $\frac{1}{n} \times \frac{1}{n-1}$. If these two should be so taken, the Probability of drawing *C* next will be $\frac{1}{n-2}$, because then only $n-2$ Letters will be remaining: Wherefore, (*by the same Cor.*) $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$ is the Probability of taking the three first according to the Order proposed: Whence, from the Manner of the Process, it is evident, that the Probability of coming out of the 4, 5, or 6 first, &c. according to the same Order, will be $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} \times \frac{1}{n-3}$, $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} \times \frac{1}{n-3} \times \frac{1}{n-4}$, or $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} \times \frac{1}{n-3} \times \frac{1}{n-4} \times \frac{1}{n-5}$, &c. respectively; and consequently that the required Probability is $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2} \times \frac{1}{n-3}$, &c. continued to p Factors, or $\frac{1}{n \times n-1 \times n-2 \times \dots}$, &c., whose Denominator consists of the same Number of Factors.

COROLLARY.

HENCE if p be supposed $=n$, or all the Letters, or Things, be

be to be taken, the above said Probability will become

$\frac{1}{1 \times 2 \times 3 \times 4, \&c. \text{ to } n}$: Therefore since there is but one Chance, or Way, for all the Letters, or Things, to come out in that Order, it follows that the whole Number of Permutations, or all the different Ways which the same Things can possibly be taken, will be the Denominator of that Fraction : Or, $1 \times 2 \times 3 \times 4, \&c.$ continued to as many Factors as there are Things, (*Def. I.*)

E X A M P L E.

SUPPOSE it were required to find the Number of Changes on 7 Bells; then taking the first 7 Factors of $1 \times 2 \times 3 \times 4, \&c.$ and multiplying them together; there comes out 5040, for the Number that was to be found.

P R O B L E M III.

IF out of any given Number (n) of Things, as A, B, C, D, E, F, &c. a Person be to take a given Number (p) of them, as it happens; what is the Probability that the Things so taken shall be the (p) first of the foregoing Order, as A, B, C, and D, &c.

S O L U T I O N.

Let the Things be taken one by one; then because, in the Whole, there are p assigned, the Probability of drawing one of these first will be $\frac{p}{n}$. If one of those should be drawn, then there remaining $n-1$, of which $p-1$ will be of the Assigned ones, the Probability of drawing one of these next would be $\frac{p-1}{n-1}$; wherefore, (*by Cor. to Prob. I.*) $\frac{p}{n} \times \frac{p-1}{n-1}$,

C

will

will be the Probability that the two first taken shall be both of the Assigned ones. If both those should be so, the Probability of taking another of the Assigned ones next, it is obvious, would be $\frac{p-2}{n-2}$; therefore (*by the same*) the Probability of all the three first being of the Assigned ones is $\frac{p}{n} \times \frac{p-1}{n-1} \times \frac{p-2}{n-2}$: Whence, from the Method of the Process, it is manifest that the Probability sought will be $\frac{p}{n} \times \frac{p-1}{n-1} \times \frac{p-2}{n-2} \times \frac{p-3}{n-3} \times \frac{p-4}{n-4}$, &c. continued to p Factors; Or, which is the same, $= \frac{p \times p-1 \times p-2 \times p-3, \&c.}{n \times n-1 \times n-2 \times n-3, \&c.}$ where the Numerator and Denominator consist each of p Factors: But the Factors in the Numerator forming an Arithmetical Progression, whose greatest Term is p , and common Difference 1, and the Number of Terms being p , the least Term or Factor must necessarily be $=1$, and the said Numerator in an inverted Order, $1 \times 2 \times 3 \times 4$, &c. to p ; and consequently the Probability $= \frac{1 \times 2 \times 3 \times 4}{n \times n-1 \times n-2 \times n-3}, \&c.$

C O R O L L A R Y.

IF the p Things, drawn one by one, as above, be mixed again among the rest; and a second Person afterwards draws, at random, an equal Number, p , at once, it is manifest, that the Probability of his taking the very same, or any other, p , Assigned ones, will be $\frac{1 \times 2 \times 3 \times 4}{n \times n-1 \times n-2 \times n-3}, \&c.$ the same as above; and therefore since there is, here, only one Chance or Way for taking the said p assigned Things, this one Chance must be to the whole Number of Chances, or all the

the several Ways that p Things can be so taken in n Things, as the Num. of the above Fraction to its Denominator, by *Def. I.* Wherefore the Number of Combinations, or different Ways by which n Things may be taken, p by p , is equal to $\frac{n \times n - 2 \times n - 3}{1 \times 2 \times 3}$, &c. to p Factors, or $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. continued to as many Factors as there are Things in one Combination.

EXAMPLE.

LET it be required to find how many different Pairs, or Combinations, of two, can be made with four Aces. Here taking $\frac{n}{1} \times \frac{n-1}{2}$ the two first Factors of the general Expression, (because two are the Things in one Combination) and writing four instead of n , we have $\frac{4 \times 4-1}{1 \times 2} = 6$ for the Number that was to be found.

PROBLEM III.

THERE are several Sorts of Things, or Letters, as a a a, b b b b b, c c c c, d d d d d d d, to be mixed promiscuously together, in order to be taken one by one, as it happens: What is the Probability that any given Number of these Sorts come out in the very Order here exhibited, i. e. that all the a's come out first, all the b's next, &c. &c.

SOLUTION.

SINCE the Number of Letters of all Sorts is here 20, and that of the first Sort 3; the Probability that these shall be all taken before a Letter of any other Sort, is $\frac{3}{20} \times \frac{2}{19} \times \frac{1}{18}$, (by the last Prob.) If these should be so taken, the

Pro-

Probability of drawing the *b*'s next in Order, it is manifest, would be $\frac{1}{17} \times \frac{2}{16} \times \frac{3}{15} \times \frac{4}{14} \times \frac{5}{13}$ (*by the same*): Wherefore the Probability that both *a*'s and *b*'s shall be taken in Order will be $\frac{1}{20} \times \frac{2}{19} \times \frac{3}{18} \times \frac{4}{17} \times \frac{5}{16} \times \frac{6}{15} \times \frac{7}{14} \times \frac{8}{13}$ (*by Prob. I.*) In like manner the Probability of taking the three first Sorts in Order, will be had $\frac{1}{20} \times \frac{2}{19} \times \frac{3}{18} \times \frac{4}{17} \times \frac{5}{16} \times \frac{6}{15} \times \frac{7}{14} \times \frac{8}{13} \times \frac{9}{12} \times \frac{10}{11} \times \frac{11}{10} \times \frac{12}{9}$; which is also the Probability of taking all the four Sorts in Order, because when the first three of them are so taken, the last must follow of course.

C O R O L L A R Y.

If the Number of Letters, or Things, of the first Sort be *p*, of the second Sort *q*, of the third *r*, &c. and if *n* be the whole Number of Things of all Sorts: It is evident by Inspection, from the foregoing Process, that the Probability that any Number of those Sorts will come out in the proposed Order is

$$\frac{1 \times 2 \times 3, \&c. \text{ to } p, \times 1 \times 2 \times 3, \&c. \text{ to } q, \times 1 \times 2 \times 3, \&c. \text{ to } r, \&c.}{n \times n - 1 \times n - 2 \times n - 3 \times n - 4 \times n - 5 \times n - 6 \times n - 7 \times n - 8 \times n - 9, \&c.}$$
 where the Numerator consists of as many Series as there are Sorts to be taken in Order, and the Denominator of as many Factors as there are in all those Series.

P R O B L E M V.

Supposing the Proportion of the Chances for the Happening of an Event to the Chances for its Failing be as a to b, in any one Tryal, and (n) be the Number of Tryals: To find the Probability that the Event happens precisely (p) times in those Tryals.

S O L U T I O N.

SINCE the Probability of Happening of the Event at any

one

one assigned Tryal is $\frac{a}{a+b}$, and that of the contrary $\frac{b}{a+b}$; the Probability that it shall happen all the p first Tryals, and fail all the rest, will therefore (*by Cor. to Prob. I.*) be $\frac{a^p}{a+b|^p} \times \frac{b^{n-p}}{a+b|^{n-p}} = \frac{a^p b^{n-p}}{a+b|^n}$: But as there is the same Probability for its Happening any other p assigned Tryals, and Failing the rest, it is manifest that as many different Ways as p Things can be taken or combined in n Things, just so often ought the said Probability $\frac{a^p b^{n-p}}{a+b|^n}$ be repeated to give the Value sought; Wherefore that Number of Ways or Combinations being $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, &c. to p Factors, (*by Prob. III.*) $\frac{a^p b^{n-p}}{a+b|^n} \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ &c. will consequently be the Value, Q. E. I.

C O R O L L A R Y.

HENCE, if p be taken equal 0, 1, 2, 3, &c. successively, the said Value will become $\frac{b^n}{a+b|^n}$, $\frac{n b^{n-1} a}{a+b|^n}$, $n \times \frac{n-1}{2} \times \frac{b^{n-2} a^2}{a+b|^n}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{b^{n-3} a^3}{a+b|^n}$, &c. for the Probability of Happening *precisely*, 0, 1, 2, 3, &c. times, respectively; where the several Terms are those of the Binomial $b+a$ raised to the Power whose Index is n , divided by that Power. And therefore if from the Binomial $b+a$ raised to the n Power, all the Terms where the Indices of a are less than p , be taken and divided by $a+b|^n$, the Quotient will, it is manifest, express the Probability that the proposed Event shall not happen so often as p times in the given Number n of Tryals: But if the remain-

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ing Terms, or those where the Indices of a are not less than p , be divided by $\overline{a+b}^n$, the Quotient will then be the Probability that it shall happen, at least, p times in those Tryals.

E X A M P L E I.

A , with four Dice, undertakes to throw one Ace and no more at the first Throw; What is the Odds against him?

By considering the Dice as thrown one by one, and the four Throws as so many Tryals, we have $n=4$, $p=1$, $a=1$, $b=5$, and therefore $\frac{nb^{n-1}a}{a+b^n}$ equal to $\frac{1 \times 5^3}{6^4} \times 4 = \frac{125}{324}$ for the Probability of throwing one Ace precisely; Wherefore that of the contrary is $\frac{199}{324}$, and the required Odds 199 to 125, or 8 to 5 nearly.

E X A M P L E II.

SUPPOSING One to throw up nine Half-pence; What is the Odds that there come up more than three Heads?

Here (by supposing as in the above Case) n will be $=9$, $a=1$, and $b=1$; Wherefore taking the 4 first Terms, $1+9+9 \times \frac{8}{2} + 9 \times \frac{8}{2} \times \frac{7}{2}$ ($=130$) of $1+1$ raised to the 9th Power, and dividing their Sum by $\overline{1+1}^9$ ($=512$) according to the Corollary, there will be $\frac{130}{512}$ for the Probability that there come not up four Heads; Therefore the Odds are exactly as 191 to 65.

E X A M P L E III.

THERE is a Lottery consisting of a great Number of Tickets, whereof the Prizes are to the Blanks in the given Proportion of 1 to 3: What is the Odds that in taking seven Tickets there shall come out two Prizes?

Suppose the Tickets to be taken one by one; and let it be first

first inquired what the Probability is that there come not out two Prizes; Then n being $=7$, $a=1$, $b=3$, and $p=2$, we shall have $\frac{b^n + nb^{n-1}a}{a+b^n} = \frac{3^7 + 7 \cdot 3^6 \cdot 1}{1+3^7} = \frac{3645}{4547}$ for the Answer; and therefore the required Odds well be as 4547 to 3645. But it must be observed that this Proportion is not exactly true, as in the foregoing Cases; for here the Ratio of a to b , or the Probability of drawing a Prize, will not every Tryal be exactly the same, but greater or lesser according as a greater or lesser Number of Blanks has been before taken; whereas in Dice, &c. the Probability of throwing a given Face, &c. ever continues the same. However when the Number of Tickets is large, the Alteration of the Ratio $\frac{a}{b}$ being then small, this Method of Solution will be sufficiently near the Truth; but in other Cases, or where a rigid Accuracy is insisted on, it will be best to have reference to the 8th Problem.

N. B. That an Expression placed in a Parenthesis, immediately after any Product or Series, is every where put to shew the Number of Factors, or Terms, to which that Product or Series is to be continued; As (p) at the End of $n \times n - 1 \times n - 2 \times n - 3$, or $n \times n - 1 \times n - 2 \times n - 3$ (p) shews that the Product $n \times n - 1 \times n - 2$, &c. is to be continued to as many Factors as there are Units in (p) .

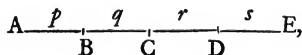
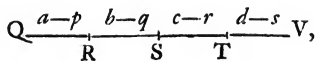
PROBLEM VI.

THERE is a given Number of each of several Sorts of Things, (of the same Shape and Size); as (a) of the first Sort, (b) of the second, &c. put promiscuously together; out of which a given Number (m) is to be taken, as it happens; To find the Probability that there shall come out precisely a given

given Number of each sort, as (p) of the first, (q) of the second, (r) of the third, &c.

SOLUTION.

IN order to render the Solution of this Problem as easy as may be, let the Things of the first Sort be selected from the rest, and of those conceive the given Number p to be placed between A and B, or in the first Division of the line AE, and the rest between Q and R, or in the first Division of QV: In like Manner, let the Things of the second and third Sorts be selected



and be disposed of after the same Method in the second and third Divisions of those Lines respectively: And let us first inquire how many Combinations may be made of m Things in the Line AE, by mutually changing, one by one, the Things in that Line for those in the other Line QV, under this Restriction, that in every Combination the Number of Things of each Sort shall still continue the same: Then because the same Number of Things of each Sort, or in each Division of both Lines, is to be preserved, a Thing of the first, second, or third Divisions of the lower must always be changed for one in the corresponding Division of the upper; and therefore it is manifest, that as many Ways as p Things can be taken in a Things, which is $\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} (p)$, *Prob. 3.* so many Combinations may be made in the Line AE, or QV, without changing any one of the Things besides those in the first

first Divisions of those Lines : And for the same Reasons it is evident, that the Number of Combinations in each of the same Lines, without changing any Thing besides those in the second or third, &c. Divisions, is $\frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3}$, (*q*) or $\frac{c}{1} \times \frac{c-1}{2} \times \frac{c-2}{3}$ (*r*), &c. respectively.

But since $\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3}$ (*q*), the said Combinations in the first Division, may be repeated every time a Thing of the second Division is changed without altering any thing in the succeeding Divisions, $\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3}$ (*p*) into $\frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3}$ (*q*) must consequently be the Number of all the Combinations that can be produced by changing the Things of the first and second, without affecting those of the other Divisions. But this Number of Combinations may in like manner be repeated every time a Thing of the third Division is changed; and therefore all the Combinations that can be produced under the proposed Restriction, without changing any of the Things besides those of the three first Divisions, will be $\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3}$ (*p*) in. $\frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3}$ (*q*) in. $\frac{c}{1} \times \frac{c-1}{2} \times \frac{c-2}{3}$ (*r*); from whence the Process and Law of Combination are manifest. Wherefore, having now found all the possible Ways that the Things can be taken to have the proposed Numbers of each Sort, we are next to see how many Ways the same Number (*m*) of Things may be had without any Restriction. This Number of Ways, if *n* be put $= a + b + c$, &c. the whole Number of Things, will be $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$ (*m*) (by the

E

afore-

aforenamed Problem): And therefore

$$\frac{\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} (p) \text{ in. } \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} (q) \text{ in. } \frac{c}{1} \times \frac{c-1}{2} \times \frac{c-2}{3} (r)}{\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} (m)}$$

must consequently be the Value. Q. E. I.

C O R O L L A R Y I.

HENCE if the Number of Things of each Sort be equal, and those proposed to be taken of each also equal, and w be put for the Number of Sorts, the Probability, it is manifest,

will become
$$\frac{\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} (p)^w}{\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} (m)^w}.$$

C O R O L L A R Y II.

WHEN there are only two Sorts of Things, then c, d, r, s , &c. become $=0$, $a+b=n$, $p+q=m$, and

$$\frac{\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} (p) \text{ in. } \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} (q)}{\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} (m)}$$
 equal to the required

Value in that Case; which, when it is proposed that no one of the first ^{of the} two Sorts shall be taken, will become $\frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2} \times \frac{b-3}{n-3} (m)$, because then p is $=0$, and $q=m$.

E X A M P L E I.

IN a Lottery consisting of 99 Tickets, whereof there are two Prizes of 1000*l*. and five of 100*l*. What is the
Pro-

Probability that in taking 6 Tickets there shall be just one Prize of each of those two Sorts? Here n being $=99$, $m=6$, $a=2$, $b=5$, $c=92$, $d=0$, &c. $p=1$, $q=1$, $r=4$, $s=0$, &c. the Value fought, by the general Theorem, will be equal to $\frac{\frac{2}{1} \times \frac{5}{1} \times \frac{92}{1} \times \frac{92}{1} \times \frac{92}{1} \times \frac{92}{1}}{\frac{99}{1} \times \frac{98}{1} \times \frac{97}{1} \times \frac{96}{1} \times \frac{95}{1} \times \frac{94}{1}} = \frac{23 \times 25 \times 89 \times 91}{19 \times 22 \times 49 \times 94 \times 97} = \frac{4656925}{186760876}$, or $\frac{1}{40}$ nearly.

E X A M P L E II.

SUPPOSE One to draw four Cards out of the whole Pack; and let it be required to find the Odds that there shall not come out just one of each Sort, as one Heart, one Diamond, &c. Because, n , the whole Number of Things (or Cards) is here $=52$, those taken (or m) $=4$; (w) the Number of Sorts $=4$; a , b , c and d , each $=13$; and p , q , r and s each $=1$: By Cor. I. we have $\frac{13^4}{1 \times 1 \times 1 \times 1} = \frac{2197}{10625}$ for the Probability of Happening; therefore that of the contrary is $\frac{18628}{2197}$, and the required Odds as 18628 to 2197, or nearly as 8 to 1.

E X A M P L E III.

LET there be a Heap of twenty Cards, wherein are seven Diamonds, six Hearts, four Spades, and three Clubs; To find the Probability that in drawing eight of them, at a Venture, there shall come out just three Diamonds and two Hearts.

Here, because the Answer is restrained only to the taking of three Diamonds, two Hearts, and three other Cards that are neither Diamonds nor Hearts, these last, or all the Black ones, in respect to the general Theorem, may, it is manifest, be considered as one Sort. And we shall have, $a=7$, $b=6$, $c=7$, $n=20$, $m=8$, $p=3$, $q=2$, $r=3$; and therefore

$\frac{1}{2} \times$

$$\frac{7 \times \frac{6}{2} \times \frac{5}{3} \times \frac{4}{4} \times \frac{3}{5} \times \frac{2}{6} \times \frac{1}{7} \times \frac{0}{8}}{\frac{20}{1} \times \frac{19}{2} \times \frac{18}{3} \times \frac{17}{4} \times \frac{16}{5} \times \frac{15}{6} \times \frac{14}{7} \times \frac{13}{8}} = \frac{35 \times 35}{17 \times 19 \times 26} = \frac{1225}{3978} \text{ the Probability:}$$

Wherefore the Odds is as 2755 to 1225, or as 9 to 4 nearly.

EXAMPLE IV.

THERE is a Lottery consisting of 10,000 Tickets, among which are three particular Prizes; What is the Odds that a Person in taking 2000 of them shall not have all those Prizes? As it matters not, here, whether there be other Prizes in the Lottery besides these three, the other 9997 Tickets may be all considered as Blanks; and then the Probability of taking the three Prizes with 1997 of these Blanks, (*by the Theorem in Cor. II.*) is $= \frac{9997 \times 9995 \times 9993}{10000 \times 9998 \times 9996} \&c. \text{ to } 1997 \text{ Factors}$: But as this

Expression, by reason of the great Multitude of Factors it involves, must be impracticable without a proper Method of Reduction, and as this will always be the Case where the Number of Things taken is very large, it may not be improper to shew here how the Theorem itself may in those Cases be contracted. In order thereto, equally multiplying both Numerator and Denominator by $1 \times 2 \times 3 \times 4 \dots (m)$, our said Theorem (*Vid. Cor. II.*) will, it is manifest, first become $\frac{a \times \frac{a-1}{2} \times \frac{a-2}{2} \dots (p)}{1 \times 2 \times 3 \times 4 \dots (m)}$ into $b \times b - 1 \times b - 2 \dots (q)$ into $m \times m - 1 \times m - 2 \dots (p)$;

where, by breaking the Denominator into two Parts, so that the first Factor of the latter may be $= b$, it will next stand thus, $\frac{a \times \frac{a-1}{2} \times \frac{a-2}{3} \dots (p)}{1 \times 2 \times 3 \dots (a)}$ in $b \times b - 1 \times b - 2 \dots (q)$ in $m \times m - 1 \times m - 2 \dots (p)$;

where equally dividing by $b \times b - 1 \times b - 2 \dots (m - a)$ it becomes

$\frac{1}{1} \times$

$$\frac{a \times \frac{a-1}{2} (p) \text{ in. } a+b-m-1 (a+q-m) \text{ in. } m \times m-1 (p)}{n \times n-1 \times n-2 \times n-3 \times n-4 (a)}$$

where putting k for $a+b-m$ = the Things remaining after (m) the proposed Number is taken, and for $a+q-m$ its equal $a-p$, it will, at length, be reduced to

$$\frac{\frac{a \times \frac{a-1}{2} \times \frac{a-2}{3} (p) \text{ in. } k \times k-1 \times k-2 (a-p) \text{ in. } m \times m-1 \times m-2 (p)}{n \times n-1 \times n-2 \times n-3 \times n-4 \times n-5 \times n-6 (a)}}$$

equal to the Probability of taking precisely p Things of that Sort whose Number is a : By help of which it will now be easy to proceed in the Prosecution of our Example; for here n being = 10000, $m=2000$, $k=8000$, $a=3$, and $p=3$, the said Expression becomes $\frac{\frac{1}{1} \times \frac{2}{2} \times \frac{1}{3} \times 2000 \times 1999 \times 1998}{10000 \times 9999 \times 9998} =$

$\frac{1997001}{249925005}$ equal to the Probability of taking all the three Prizes; Wherefore the required Odds is as 247928004 to 1997001, or as 124 to 1 nearly.

E X A M P L E V.

THE same Things being supposed as in the last Case; Let it be required to find the exact Odds that there comes out one or more Prizes?

In this Case, it is manifest, the Answer may be obtained by seeking severally the Probabilities of taking one, two, and three Prizes, and adding them together, &c. But by first making the Probability of taking no Prize at all the Subject of our Inquiry, it will be easily had at one Operation: For then p being = 0, the general Expression (contracted as above) becomes $\frac{8000}{10000} \times \frac{7999}{9999} \times \frac{7998}{9998} = \frac{127952004}{249925005}$; and there-

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fore the Odds are as 127952004 to 121973001, that there comes out no Prize.

PROBLEM VII.

Supposing a great, but given Number of each of two Sorts of Things to be put promiscuously together; To find how many must be taken out of the Whole, to make it an equal Chance that they shall all come out of one given Sort.

SOLUTION.

If n be put for the whole Number of Things, b those of the given Sort, and m the Number required; it is manifest from Cor. II to the last Prob. that the Probability of taking all the m Things of that Sort will be $\frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2} (m)$;

which, by the Question, must here $= \frac{1}{2}$; but $\frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2}$ being nearly equal; ~~hence b and n are nearly equal Numbers,~~

$\frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2} (m) (= \frac{1}{2})$ will be $\frac{b}{n} \times \frac{b}{n} \times \frac{b}{n} (m)$, or $\frac{b^3}{n^3}$ very nearly; whence in Logarithms m into $\text{Log. } b - \text{Log. } n = -\text{Log. } 2$, and $m = \frac{\text{Log. } 2}{\text{Log. } n - \text{Log. } b}$. Q. E. I.

EXAMPLE.

IN a Lottery consisting of 100,000 Tickets, in which the Blanks are to the Prizes as 9 to 1, how many ought one to take, to make it an equal Chance that there shall come out one or more Prizes? Or, which is the same, that there shall be no one of them Prizes? Here n being 100,000, and b 90,000, the general Expression becomes $\frac{.30103}{5-4.95424} = 6.58$; which not coming out a whole Number, shews there can be no exact Equality

Equality of Chance in this Case, 6 being too small, and 7 too great a Number.

PROBLEM VIII.

THE same being supposed as in the last Problem; To find the Probability that in taking a given Number (m) of those Things, there shall not come out a given Number (p) of one Sort.

SOLUTION.

THIS Problem may be solved in any Case, from Cor. II. to Prob. VI. by finding at several Operations the Probability of taking precisely all the inferior Numbers to p the proposed one. But these Operations may be very much contracted; for if (as before) a be put for the Things of one Sort, b those of the other, and n the whole Number of both Sorts, it is manifest from thence, that

$$\frac{\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} \times (p) \text{ in } \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} (m-p)}{\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} (m)} \text{ will be the Pro-}$$

bability of taking p Things precisely of that Sort whose Number is a . Therefore if p be supposed equal to 0, 1, 2, 3,

&c. successively, and A be put $= \frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2} (m) =$ the

Probability of taking none of the said Sort, and $b-m+1=b$;

then will $A, \frac{m}{b} \times \frac{a}{b+1} A, \frac{m}{b} \times \frac{m-1}{b+1} \times \frac{a}{1} \times \frac{a-1}{2} A, \frac{m}{b} \times \frac{m-1}{b+1} \times \frac{m-2}{b+2} \times \frac{a}{1} \times$

$\frac{a-1}{2} \times \frac{a-2}{3} A$, &c. be the respective Probabilities of drawing

0, 1, 2, 3, &c. of that Sort, precisely: Wherefore $A + \frac{m}{b} \times$

$\frac{a}{1} A + \frac{m-1}{b+1} \times \frac{a-1}{2} \times B + \frac{m-2}{b+2} \times \frac{a-2}{2} C + \frac{m-3}{b+3} \times \frac{a-3}{3} D$, &c.

con-

continued to as many Terms as there are Unites in p , will be the Probability sought; B, C, D, &c. denoting the preceding Terms.

E X A M P L E.

IN a Heap of 10 Cards whereof one half are red and the other half black; What is the Probability that in drawing 5 of them at a Venture, there shall come out 3 red ones? Or, which is the same, that there shall not come out 3 black ones? Here a being $=5$, $b=5$, $n=10$, $p=3$, and $m=5$; A or $\frac{b}{n} \times \frac{b-1}{n-1} (m)$ will be $\frac{1}{252}$, and $b (=b-m+1)$ $=1$; wherefore $A + \frac{m}{b} \times \frac{a}{1} A + \frac{m-1}{b+1} \times \frac{a-1}{2} B (p)$ becomes $\frac{1}{252} + \frac{25}{252} + \frac{100}{252} = \frac{1}{2}$; which therefore is the required Value in this Case.

P R O B L E M IX.

THE same being still supposed as in the preceding Problems, and that the said Things are to be taken one by one as it happens; To find the Probability that the (p) first shall all come out of the first Sort, and the next after of the contrary Sort.

S O L U T I O N.

THE Number of the Things of the first Sort being a , that of the second Sort b , and the whole Number of both Sorts n , the Probability of taking one of the first Sort first, will, it is manifest, be $\frac{a}{n}$; If this should happen, that is, if one of that Sort should be actually taken, then there remaining only $a-1$ of that Sort, the Probability of taking one of those next would

would be $\frac{a-1}{n-1}$; wherefore the Probability of taking both

the two first of the said Sort will be $\frac{a}{n} \times \frac{a-1}{n-1}$ (*Proposition I.*)

From the Manner of which Process it is evident that the Probability of taking all the first m of this Sort will be $\frac{a}{1} \times \frac{a-1}{n-1} \times$

$\frac{a-2}{n-2} \times \frac{a-3}{n-3} (m)$: But if these should be so taken, the Pro-

bability of taking one of the other Sort next will be $\frac{b}{n-a}$;

Wherefore $\frac{a}{n} \times \frac{a-1}{n-1} \times \frac{a-2}{n-2} (m)$ in. $\frac{b}{n-a} = \frac{b}{n} \times \frac{a}{n-1} \times \frac{a-1}{n-2} \times$

$\frac{a-2}{n-3} \times \frac{a-3}{n-4} (m+1)$ is the Value that was to be found.

PROBLEM X.

TO determine the Odds at Bowls, or other Games of the like Nature, in any Circumstance of the Play.

SOLUTION.

FIRST suppose the Players to be only A and B ; or, that there are only 2 Bowls of a Side: Then, as the Probability that either of them shall win just 1 Bowl at an End is the same as that of taking 2 Things out of 4 Things of 2 Sorts, so that the first may come out of the first Sort, and the other of the contrary; the said Probability, it is manifest, will be $\frac{2}{4} \times \frac{2}{3} = \frac{1}{3}$, by the last Problem; And, for the like Reason, the Probability of winning 2 Bowls at an End will, by the same, be had $= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{6}$. And the like may be had in any other Case when the Players are a greater Number.

This being premised; Let A want 2 of being up, and B 1; and the Value of the Thing play'd for be denoted by 1.

If A gets just 1 at the first End, of which the Probability is $\frac{1}{3}$ (*per above*) he will be in the same Circumstance with his Antagonist, and therefore entitled to $\frac{1}{3}$ the whole Stake, or, $\frac{1}{3}$; Wherefore his Expectation on that Event, while it remains uncertain, will be $\frac{1}{3}$ of $\frac{1}{3} = \frac{1}{9}$. But if he should get 2 he will be intitled to (1) the whole Stake; wherefore as the Probability of getting 2 is $\frac{1}{6}$, his Expectation on this Event will be, also, $\frac{1}{6}$, which added to the foregoing will consequently give ($\frac{1}{9}$), the total Expectation of A in the proposed Circumstance; which subtracted from (1), the whole Stake, leaves that of $B = \frac{2}{3}$; and therefore the Odds are as 2 to 1 in this Case.

Let A want 3, and B as before. Because the Probability that A gets just one at the first End, or there brings the Play to the Circumstance of the above Case, is $\frac{1}{3}$; and as his Expectation in that Circumstance will also be $\frac{1}{3}$ (*per above*) his present Expectation on that Event will therefore be $\frac{1}{9}$. But if he should get 2, he will be entitled to half the whole Stake, or $\frac{1}{2}$; which therefore multiplied by $\frac{1}{6}$, the Probability of getting 2, gives $\frac{1}{12}$ for his Expectation on this Event; Whence $\frac{1}{9} + \frac{1}{12} = \frac{7}{36}$, the Sum of these 2, must be the total Expectation of A in this Case; hence that of $B = \frac{29}{36}$, and the Odds as 29 to 7.

Let A want 3, and B 2. Now A must either get 1 or 2 at the first End; or, B 1 or 2 at the same End.

If A gets 1 he is entitled to $\frac{1}{2}$ the Stake, which multiplied by $\frac{1}{3}$, the Probability of it, is $\frac{1}{6}$, the Expectation on that Event.

If he gets 2 he is entitled to $\frac{2}{3}$, (*by the first Case*); therefore the Expectation on this Event is $\frac{1}{6} \times \frac{2}{3} = \frac{1}{9}$.

If B gets 1, A will be entitled to $\frac{2}{3}$, (*by the last Case*), and there-

therefore on this Event his Expectation is $\frac{1}{3} \times \frac{7}{10} = \frac{7}{30}$. But if B gets 2 the Expectation of A will be nothing at all; Therefore $\frac{1}{3} + \frac{1}{9} + \frac{7}{10} = \frac{17}{10}$, the Sum of those 3, is the total Expectation of A in this Case. And by proceeding on in this Manner the Expectations of the Gamesters may be determined in any other Circumstance of the Play; and Tables calculated to exhibit the Odds, not only when the Players are 2, but a greater Number. The first of the 2 following Tables shews the Odds when there are only 2 Players; and the last, the Limits of the least Odds when the Players are indefinite, the Odds in any Circumstance of the Play growing less as the Number of Players is increased.

		Truly.	Nearly.
2	1	2 : 1	2 : 1
3	1	29 : 7	4 $\frac{1}{2}$: 1
4	1	95 : 13	7 $\frac{1}{2}$: 1
5	1	601 : 47	12 $\frac{1}{2}$: 1
3	2	71 : 37	1 $\frac{1}{2}$: 1
4	2	247 : 77	3 $\frac{1}{2}$: 1
5	2	272 : 52	5 $\frac{1}{2}$: 1
4	3	607 : 365	1 $\frac{1}{2}$: 1
5	3	8507 : 3157	2 $\frac{1}{2}$: 1
5	4	13507 : 21485	1 $\frac{1}{2}$: 1

		Truly.	Nearly.
2	1	5 : 3	1 $\frac{1}{2}$: 1
3	1	23 : 9	2 $\frac{1}{2}$: 1
4	1	101 : 27	3 $\frac{1}{2}$: 1
5	1	431 : 81	5 $\frac{1}{2}$: 1
3	2	77 : 51	1 $\frac{1}{2}$: 1
4	2	175 : 81	2 $\frac{1}{2}$: 1
5	2	1535 : 513	3 : 1
4	3	601 : 423	1 $\frac{1}{2}$: 1
5	3	5311 : 2781	1 $\frac{1}{2}$: 1
5	4	18909 : 13859	1 $\frac{1}{2}$: 1

Now by the Help of these Tables the Odds may be nearly had in any intermediate Case, as follows, *viz.* In the proposed Circumstance, be it what it will, from the said Tables, find the Probabilities of winning in those two Cases for which they are calculated, and let those Probabilities be denoted by p and P respectively, and let n be the Number of Players in the Case proposed; then will $P - \frac{P-p}{n}$ be the Probability of winning in that Case, nearly.

EXAMPLE.

SUPPOSE the Number of Players to be 6; and one Side to

want

want 2 of being up, and the other 4. Then by the Tables the Odds will be as $3\frac{1}{2}$ to 1, and $2\frac{1}{2}$ to 1; and therefore the Probabilities of winning $\frac{2}{3}$, and $\frac{2}{5}$ respectively, (in those Cases for which the Tables are calculated under the same Circumstance); Wherefore P is here $=\frac{6}{10}$, and $p=\frac{2}{5}$, and therefore $P - \frac{P-p}{\frac{1}{2}n} = \frac{147}{199} = \frac{2}{3}$ nearly = the Probability of winning in the Case proposed; whence the Odds are as 5 to 2 nearly.

PROBLEM XI.

TWO Persons (or Parties), A and B, play together with (n) Bowls a Side; the Skill of A to that of B, or the Odds that any assigned Bowl of A is nearer to the Jack than any assigned Bowl of B, is in the given Ratio of (a) to (b); To find the Probability that A has of getting a given Number (p) or more, and also that of getting that Number precisely at any End assigned.

SOLUTION.

SINCE the Odds, or Chances, that any assigned Bowl of A comes nearer the Jack than any assigned Bowl of B are as a to b ; each Bowl of A may be supposed to contain a Chances and each one of B, b Chances; and then the Number of Chances in all the Bowls of A being na , and the total Number of all the Chances $na+nb$, the Probability that some one of A's Bowls comes nearer to the Jack than any one of B's will, it is manifest, be $\frac{na}{na+nb}$, or, if ar be put $=n \times a + b$, equal to $\frac{n}{r}$. Now if 1 of A's Bowls should come nearest then, he having $n-1$ Bowls, or $a \times n - 1$ Chances remaining, the Probability of his having a second Bowl nearer to the Jack than any

any one of *B*'s would, it is manifest, be $\frac{a \times n - 1}{an + bn - a}$ or $\frac{a \times n - 1}{ra - a} = \frac{n-1}{r-1}$; Wherefore the Probability that *A* shall have 2 Bowls nearer to the Jack than any one of *B*'s is $\frac{n}{r} \times \frac{n-1}{r-1}$ (by Cor. to Prop. I.) In like Manner, the Probability that *A* shall have *p* Bowls nearer to the Jack than any one of *B*'s, will be found $\frac{n}{r} \times \frac{n-1}{r-1} \times \frac{n-2}{r-2} \times \frac{n-3}{r-3} (p)$; which is an Answer to the first Part of the Question: Now if that should happen, of which this last Expression is the Probability; then there remaining $r - p \times a$ Chances, and *nb* of them in Favour of *B*, the Probability that one of his Bowls comes next will be $\frac{nb}{r - p \times a}$; Therefore (by the same Cor.) $\frac{nb}{r - p \times a}$ in. $\frac{n}{r} \times \frac{n-1}{r-1} \times \frac{n-2}{r-2} \times \frac{n-3}{r-3} (p) = \frac{nb}{ra} \times \frac{n}{r-1} \times \frac{n-1}{r-2} \times \frac{n-2}{r-3} (p+1)$ will be the Value. Q. E. I.

Note, By help of this Theorem, and the Method of Proceeding in the foregoing Problem, the Probability, or Odds of winning the Game, may be determined in any Case whatever, where the Ratio of *a* to *b*, or the Proportion of Skill, can be ascertained.

PROBLEM XII.

A and *B* playing together with single Bowls, Coits, or Pieces, &c. the former finds by Experience that he can upon an Equality of Chance, undertake to win (*n*) Times, before his Antagonist once: What is the Proportion of Skill of the 2 Gamesters, or their Chances of winning at any assigned Tryal?

SOLUTION.

LET the required Ratio be that of a to b , or, which is to the same Effect, let $\frac{a}{a+b}$ be the Probability of A 's winning at the first Tryal; then by Cor. to Prop. I. $\frac{a^n}{a+b^n}$ will be the Probability of his winning all the n first Times, or Tryals, which \therefore by the Question must be $=\frac{1}{2}$; whence $2a^n = a+b^n$ and $a \times 2^{\frac{1}{n}-1} = b$; therefore, as $1 : 2^{\frac{1}{n}-1} :: a : b$. Q. E. I.

PROBLEM XIII.

A and B, whose Proportion of Skill, or Chances for winning any assigned Game, are as (a) to (b), play together; the former wants (p) Games of the whole Set, and the latter (q); What are their respective Probabilities of winning?

SOLUTION.

SUPPOSE the Play to be continued after the Set is out, till such time as $p+q-1$ Games are expired; and a Spectator E to wager with F that A beats his Adversary p of these Games. Then, (by Prob. V.) it is manifest the Probability of E 's winning will be

$$\frac{a^n + na^{n-1}b + n \times \frac{n-1}{2} a^{n-2}b^2 + \dots + n \times \frac{n-1}{2} a^p b^{n-p}}{a+b^n} : \text{But this}$$

is the Probability that A wins the Set; because if he gets p of those $p+q-1$ Games he must lose fewer than q of them, and therefore get p before he loses q ; and because it is evident that whatever the Chance of E may be in respect of winning and losing, that of A must be the same, and *vice versa*.

versa. And therefore, for the same Reason, the Probability

that B wins the Set must be
$$\frac{b^n + nb^{n-1}a + nx\frac{n-1}{2}b^{n-1}a^2}{a+b^n}$$

.....
$$\frac{nx\frac{n-1}{2} \&c. b^2 a^{n-2}}{a+b^n}$$
 Q. E. I.

EXAMPLE.

SUPPOSE A to want 5 of being up, B 3, and the Skill of the 2 Gamesters to be equal: Then will $p=5$, $q=3$, $a=1$, $b=1$, $n=7$, and $\frac{1+7+21+35+35}{2^7} (= \frac{80}{128})$ and $\frac{21+7+1}{2^7} (= \frac{29}{128})$, be the Probabilities of winning; therefore the Odds that B beats A are as 99 to 29.

PROBLEM XIV.

A Given Number of Gamesters, A , B and C , &c. whose Chances for winning any assigned Game are in the given Ratio of a , b , c , &c. play together; A wants p Games of the whole Set, B , q , and C , r , &c. What are their several Probabilities of winning?

SOLUTION.

RAISE $a+b+c$, &c. to the Power whose Index is equal to the least of the given Numbers p , q , r , ^{&c.} or, if there be no least, to one of the least equal ones, as \hat{p} ; and from that Power take out the Term wherein the Exponent of the Correspondent Quantity a is equal to p ; and if there be any Terms wherein the Indices of b and c , &c. are equal to q , r , &c. take those Terms also, and having divided each of the Terms so taken by $a+b+c$, &c. or b^p , which is supposed equal to it, place the several Quotients each in different Columns,

Columns, mark'd (for Distinction sake) A , B and C , in that the said Power of a may be in the Column A , that of b (if any) in the Column B , &c. And then, having proceeded thus far, multiply the remaining Terms of the said Power by $a+b+c$, &c. and from the Product select all the Terms where the Indices of the Powers of a , b , c , &c. are respectively equal to p , q , r , &c. And having divided each by b^{p+1} , place them, according to the foregoing Method, in the Columns A , B , C , &c. Let the last Remainder be multiplied in like manner by $a+b+c$, &c. and select, again, out of this Product the Terms wherein the Exponents of a , b , c , &c. are respectively equal to p , q , r , &c. and having divided each by b^{p+2} , let the Quotients be disposed of as before. And proceed on in this same Manner, repeating the Operation till all the Terms are exhausted; then the Quantities that are, at last, found in the Columns A , B , C , &c. will be, respectively, the required Probabilities of winning.

E X A M P L E.

SUPPOSE the Number of Players to be 3, as A , B and C , and they to want 1, 2, and 3, respectively.

Having first raised $a+b+c$ to the 1 (p) Power, and prepared 3 Columns A , B , C , I take a ($=a^1$) from the said Power, divide it by b ($=a+b+c^0$) and place the Quotient $\frac{a}{b}$ in the Column A ; then multiply the Remainder $b+c$ by $a+b+c$, and

$$\begin{array}{r}
 A \quad \frac{a}{b} + \frac{ab+ac}{bb} + \frac{2abc+acc}{b^3} + \frac{3abc^2}{b^4} \\
 \hline
 B \quad \frac{bb}{bb} + \frac{2bbc}{b^3} + \frac{3bcc}{b^4} \\
 \hline
 C \quad \frac{c^3}{b^3} + \frac{3bc^2}{b^4} \\
 \hline
 \end{array}$$

from

from the Product $ab+ac+bb+2bc+cc$, take the Terms $ab+ac$, bb , divide each by bb , and place the 2 Quotients $\frac{ab+ac}{bb}$, $\frac{bb}{bb}$ in the Columns A and B ; multiply the last Remainder $2bc+cc$, by $a+b+c$; from the Product $2abc+acc+2bbc+3bcc+c^2$ select $2abc+acc$, $2bbc$, and c^2 , divide each of them by b^2 , and dispose of the Quotients as before; lastly, I multiply the new Remainder $3bcc$ by $a+b+c$, divide each of the 3 Terms in the Product by b^2 , and place the Quotients in the Columns A , B , and C as above, and then find the total Values in these 3 Columns to be $\frac{a}{b} + \frac{ab+ac}{bb} + \frac{2abc+acc}{b^2} + \frac{3abcc}{b^2}$, $\frac{bb}{bb} + \frac{2bbc}{b^2} + \frac{3b^2c^2}{b^2}$, and $\frac{c^2}{b^2} + \frac{3bc^2}{b^2}$; these are respectively equal to the Probabilities required; which when a , b and c are equal, will therefore become $\frac{19}{27}$, $\frac{6}{27}$, and $\frac{2}{27}$.

Note, *The above Method of Solution is only a compounded Case of Cor. to Prob. I. and therefore to such as understand that well, the Reasons of this will not be difficult.*

PROBLEM XV.

THERE being (a) Chances for the Happening of an Event, and (b) Chances for the Contrary at any assigned Tryal; In how many Tryals may one undertake that the said Event shall happen (r) times?

SOLUTION.

LET n be the Number sought; then

$$\frac{b^n + nb^{n-1}a + n \times \frac{n-1}{2} b^{n-2}a^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} b^{n-3}a^3 (r)}{a+b|^n} \text{ being the}$$

I

the Probability that Event shall not happen r times in n Tryals (*by Prob. V.*), must therefore by the Question be $= \frac{1}{2}$; whence $b^n + nb^{n-1}a + n \times \frac{n-1}{2} b^{n-2}a^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} b^{n-3}a^3 (r) = \frac{a+b}{2}^n$; where substituting pb for a we have $1 + np + n \times \frac{n-1}{2} p^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} p^3 (r) = \frac{1+p}{2}^n$; from whence by the common Methods of converging or infinite Series the Value of n may in any Case be determin'd.

Suppose $a=b$, or $p=1$; then since $1 + np + n \times \frac{n-1}{2} p^2 (r)$ the first half of the Terms of the Binomial $\frac{1+p}{2}^n$ expanded in a Series is equal to the rest of the Terms, or half the whole Power, the whole Number of Terms must be $= 2r$, and therefore n the Index of the Power $= 2r-1$, which is the required Value in this Case.

Let $\frac{a}{b}=p$ be very small; then because in this Case n must be very great, the Numbers 1, 2, 3, &c. in the Factors $n-1$, $n-2$, $n-3$, &c. may be rejected as inconsiderable in respect of n , which done, the Equation becomes $1 + pn + \frac{n^2 p^2}{2} + \frac{n^3 p^3}{2.3} + \frac{n^4 p^4}{2.3.4} (r) = \frac{1+p}{2}^n$; where, by substituting $r+v$ instead of pn , we get $1 + r + v + \frac{r+v}{2} + \frac{r+v}{2 \times 3} + \frac{r+v}{2 \times 3 \times 4} (r) = \frac{1+p}{2}^{\frac{r+v}{p}}$ or, by bringing the several Powers of $r+v$ into simple Terms, it will be

$$\left| \begin{array}{l} 1, \text{ in } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3} (r) \\ v \text{ in } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3} (r-1) \\ \frac{v^2}{2} \text{ in } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3} (r-2) \\ \frac{v^3}{2.3} \text{ in } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3} (r-3) \end{array} \right| = \frac{1+p}{2} \frac{r+v}{p}$$

Put $e=1+r+\frac{r^2}{2}+\frac{r^3}{2.3} (r)$, $f=1+r+\frac{r^2}{2}+\frac{r^3}{2.3} (r-1)$ $g=1+r+\frac{r^2}{2}+\frac{r^3}{2.3} (r-2)$, &c. and $s=\text{hyperb. Log. of } 2e$; then it

will become $e+fv+\frac{gv^2}{2}+\frac{hv^3}{2.3}+\frac{jv^4}{2.3.4}+\frac{kv^5}{2.3.4.5}$ &c. $=\frac{1+p}{2} \frac{r+v}{p}$

or in Logarithms, $s+\frac{fv}{e}+\frac{ge-ff}{2ee} \times vv+\frac{h}{6e}-\frac{fg}{2ee}+\frac{f^2}{3e^2} \times v^2$ &c.

$=r+v$ in $1-\frac{p}{2}+\frac{p^2}{3}$, &c. $=r+v$ very nearly, because p is

supposed very small; wherefore by Reduction $v+\frac{ff-ge}{2exe-f} \times$

$\frac{3fg-eb+\frac{2f^3}{e}}{6exe-f} \times v^2$ &c. $=ex \frac{s-r}{e-f}$; whence by putting these known Coefficients of the Powers of v , equal to 1, B , C , &c. respectively, and $ex \frac{s-r}{e-f} = z$, we have $v=z-Bz^2+\frac{2B^2-C \times z^3+5BC-5B^3-D \times z^4}{2B^2-C \times z^3+5BC-5B^3-D \times z^4}$ &c. and therefore $pn (=r+v) = r+z-Bz^2+\frac{2B^2-C \times z^3}{2B^2-C \times z^3+5BC-5B^3-D \times z^4}$ &c. or, because the Series converges very swift, $pn=r+z$ nearly; wherefore $\frac{r+z}{p}$, &c. is the required Value in this Case.

Hence if $r=1$, then will $e=1$, $f=0$, &c. $s=\text{hyp. Log. } 2=$
 $.69314$, $z=-.30686$, $pn=.69314$, and $n=\frac{.69314}{p}$.

If $r = 2$; then $e = 3$, $f = 1$, $g = 0$, &c. $s = \text{hyp. Log. } \frac{6}{\text{sum of the}}$
 $= 1.7917$, $z = -.31236$, and $pn = 1.6784$; but $r + z$, the 2
 first Terms of the Series, is $= 1.6876$; and therefore greater
 than the Truth only by $.0092$.

Lastly, if r be taken equal to,

3 , 4 , 5 , 6 , 7 , 8 , 9 and 10 succes.
 then will | 2.6743 | 3.6720 | 4.6702 | 5.6702 | 6.6698 | 7.6695 | 8.6693 | 9.6692 | be the
 Value of pn respectively; from whence it appears that (n)
 the required Value, in all Cases where p is very small, will be
 $= \frac{r-.3}{p}$ nearly, or $pn = r - \frac{3}{10}$. Now from this, and the fore-
 going Conclusion, where p was supposed $= 1$, and pn found
 $= 2r - 1$, the following Theorem is deduced; by help of
 which, the Value of n in any intermediate Case may be
 obtained, it being always $= \frac{1}{2} \times r - .3 + r - \frac{7}{10} = 1.50$ the Number that was to be
 found.

E X A M P L E I.

In how many Throws with 3 common Dice may one un-
 dertake to throw the 3 Aces.

The Number of Chances for Failing at any Tryal being
 215 , and for happening only one, $\frac{1}{8}$ will therefore be
 $= 215$ and $\frac{1}{8} \times r - .3 + r - \frac{7}{10} = 1.50$ the Number that was to be
 found.

E X A M P L E II.

IN a Lottery consisting of a great Number of Tickets,
 where the Blanks are to the Prizes as 50 to 1; To find how
 many a Person ought to take to expect 5 Prizes?

Here $50 \times 5 - .3 + 5 - .7 = 231$, is the Answer.

E X

E X A M P L E III.

SUPPOSE a Lottery like the foregoing, where the Blanks are to the Prizes, as 3 to 1; To find how many must be taken to expect 8 Prizes?

Here r being 8, and $\frac{1}{2} = 3, \frac{1}{2} \times r - .3 + r - .7$ becomes 30.4; therefore 30 or 31 is the Answer.

E X A M P L E IV.

IN how many Throws, with a single Die, may a Person undertake to throw either the Ace or Duce?

Here r being = 1, $\frac{1}{2} = 2, \frac{1}{2} \times r - .3 + r - .7$ will be 1.7, differing from the true Value 1.709 (*found from the Theorem in Prob. VII.*) by .009 only.

P R O B L E M XVI.

Supposing a given Number (n) of Letters a, b, c, d, e, f, &c. or Things represented by them, to be placed in Order, and that a Person draws them one by one, as it happens, and lays them down in the Order they are taken; To find the Probability that any (p) assigned Letters shall have the very same Places in the second as in the first Order, and (m) other assigned ones, at the same time, all different Places from what they have in that Order.

S O L U T I O N.

THE Probability that b happens not in the second Place of the second Order, and that, that a happens in the first Place, and b out of the second Place of this Order, are $1 - \frac{1}{n}$, and $\frac{1}{n} - \frac{1}{n \times n - 1} \left(= \frac{1}{n} \times \frac{n-2}{n-1} \right)$ respectively (*by Prob. I.*) And

therefore if the last of these be subtracted from the former, or, the Probability that b shall not come into the second Place, without any Restriction of having a either in or out of the first Place, the Remainder $1 - \frac{2}{n} + \frac{1}{n \times n - 1}$, it is manifest, will be the Probability that neither b shall come in the second Place, nor a in the first; since a must necessarily be either in the first Place or out of it; and the same is, also, the Probability that any other 2 assigned Letters shall happen in different Places from what they possess in the first Order. Wherefore if a should be the first taken, of which the Probability is $\frac{1}{n}$, then because there would remain only $n-1$ Letters, the Probability of b and c both happening out of their Places (by substituting $n-1$ for n , &c. in the said Expression) would, it is manifest, be expressed by $1 - \frac{2}{n-1} + \frac{1}{n-1 \times n-2}$; therefore this drawn into $\frac{1}{n}$ is $\frac{1}{n} - \frac{2}{n \times n - 1} + \frac{1}{n \times n - 1 \times n - 2}$ equal to the Probability that a shall fall in, and b and c out of their Places; which being subtracted, in like manner, from $1 - \frac{2}{n} + \frac{1}{n \times n - 1}$, the Probability of b and c both falling out of their Places, without farther Restriction, leaves $1 - \frac{5}{n} + \frac{3}{n \times n - 1} - \frac{1}{n \times n - 1 \times n - 2}$ for the Probability that b , c and a , or any other 3 assigned Letters shall all happen out of their Places. And therefore if a should first happen to be taken, as there would then be only $n-1$ Letters left, the Probability that b , c and d would all happen out of their Places is $1 - \frac{5}{n-1} + \frac{3}{n-1 \times n-2} - \frac{1}{n-1 \times n-2 \times n-3}$; this, therefore, multiplied by $\frac{1}{n}$, the Probability

bility of taking a first, will be the Probability that a shall happen in its Place, and b, c and d all out of their Places; whence by subtracting, as above, the Probability of a, b, c and d , or any other 4 assigned Letters, falling all out of their Places appears to be $1 - \frac{4}{n} + \frac{6}{n \times n - 1} + \frac{4}{n \times n - 1 \times n - 2} +$

$\frac{1}{n \times n - 1 \times n - 3}$: And from hence the Manner of the Process, and Law of Continuation are manifest; the Numerators being the Unciæ of the Power of a Binomial whose Exponent is equal to the Number of Letters excluded their Places; and the Denominators 1, n , $n \times n - 1$, $n \times n - 1 \times n - 2$, &c. And therefore the Probability that all the m assigned

ones shall happen out of their Places is $1 - \frac{m}{n} + \frac{m \times \frac{m-1}{2}}{n \times n - 1} -$

$\frac{m \times \frac{m-1}{2} \times \frac{m-2}{3}}{n \times n - 1 \times n - 2} + \frac{m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}}{n \times n - 1 \times n - 2 \times n - 3} (m+1)$: But the

Probability of any p assigned Letters, as a, b, c , &c. falling in their Places is $\frac{1}{n \times n - 1 \times n - 2 (p)}$ (by Prob. II.) ; and if these Letters should be first so taken, the Probability then of m other assigned ones falling all out

of their Places, will be $1 - \frac{m}{n-p} + \frac{m \times \frac{m-1}{2}}{n-p \times n-p-1} -$

$\frac{m \times \frac{m-1}{2} \times \frac{m-2}{3}}{n-p \times n-p-1 \times n-p-2} (m+1)$, per above, (n the Number

of Letters here becoming $n-p$): Therefore $\frac{1}{n \times n - 1 \times n - 2 (p)}$

in. $1 - \frac{m}{n-p} + \frac{m \times \frac{m-1}{2}}{n-p \times n-p-1} - \frac{m \times \frac{m-1}{2} \times \frac{m-2}{3}}{n-p \times n-p-1 \times m-p-2}$
 $(m+1)$ must consequently be the Value. Q. E. I.

COROLLARY I.

IF $n-p$ be $=m$, or, the p assigned Letters be proposed to be taken in, and all the rest out of, their Places, the Probability becomes $\frac{1}{n \times n-1 \times n-2} (p)$ in $1-1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} (m+1)$; which, when m is a large Number, is equal to $\frac{0.367878}{n \times n-1 \times n-2 \times n-3 \times n-4} (p)$ very nearly.

COROLLARY II.

IF a given Number p be to be taken in their Places, and the rest out of their Places, without farther Restriction, then the Probability of taking any p assigned ones in their Places, and the rest otherwise, being $\frac{1}{n \times n-1 \times n-2} (p)$ in $1-1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} (m+1)$; it is manifest, that as many different Ways as p Things can be assigned, or taken in n Things, which is $n \times \frac{n-1}{2} \times \frac{n-2}{3} (p)$, so many times ought the said Quantity be repeated to give the Probability in this Case; which therefore will be $\frac{1}{1.2.3.4} (p)$ in. $1-1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} (m+1)$; or, when m is large $= \frac{0.367878}{1.2.3.4.5} (p)$ nearly.

COROLLARY III.

HENCE if b be put $=0.367878$, and p be taken $=0, 1, 2,$

2, 3, &c. ſucceſſively, the ſaid Probability, will, it is evident, become $b, b, \frac{b}{2}, \frac{b}{2.3}, \frac{b}{2.3.4}$, &c. reſpectively equal to the Probability of taking preciſely 0, 1, 2, 3, &c. Letters according to their Places; wherefore if $b+b+\frac{b}{2} (p)$ be ſubtracted from

Unity the Remainder $1-b \times 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} (p)$ will be the Probability, very nearly, that p , or a greater Number of Letters, fall according to their Places in the firſt Order.

E X A M P L E I.

LET it be required to find how many Changes can be rung on 5 Bells, without any one ſtriking in its own proper Place.

Here becauſe p is = 0, $n=5$; and $m=5$, the Probability $\frac{1}{n \times n - 1 \times n - 2} (p)$ in. $1 - 1 + \frac{1}{2} - \frac{1}{2.3} (m+1)$, (as in Cor. I.) will become $1 - 1 + \frac{1}{2} - \frac{1}{2.3} + \frac{1}{2.3.4} - \frac{1}{2.3.4.5} = \frac{44}{120}$; and therefore as all the Changes on 5 Bells are 120, the required Number will be 44.

E X A M P L E II.

A Perſon holding 2 Packs of Cards, draws one out of one Pack, and another out of the other, and puts them together, and does the like by 2 others, &c. repeating the Operation till all the Cards are exhausted; What is the Odds that the Cards in one or more of thoſe Couples ſhall be found the ſame?

In this Caſe, the Number (52) of Things being large, 0.367878 will, either by Cor. I. or III. be the Probability that the Cards in no one Couple ſhall be the ſame: Therefore as 0.632122 is to 0.367878, ſo is the required Odds that the Cards in one or more of the Couples ſhall be the ſame.

P R O B L E M X V I I .

Supposing a given Number (r) of each of several Sorts of Things, as aaa, bbb, ccc, ddd, eee, fff, &c. to be put together in Order, and afterwards drawn one by one at a Venture, and laid down in the Order they are taken; To find the Probability that any (p) assigned Sorts shall happen to have the same Places in the second as in the first Order, and (m) other assigned Sorts, at the same time, different Places from what they have in that Order.

S O L U T I O N .

THIS Problem is solved by the same Method of Reasoning as the foregoing; for if n be put = the total Number of Things of all Sorts, and $1 \times 2 \times 3 \times 4 \times 5 \dots (r) = s$, it is evident, from Prob. III. that $\frac{s}{n \times n - 1 \times n - 2 \dots (r)}$ will be the Probability that all the a 's are taken first; and therefore that of the contrary, or the Probability that all the Things of any assigned Sort shall not happen as they are in the first Order, must be $1 - \frac{s}{n \times n - 1 \dots (r)}$. Therefore if the a 's should be all taken first, as there would be then only $n - r$ Things left, the Probability that the b 's would not, all, come out next, or fall in their Place, must, it is evident, be $1 - \frac{s}{n - r \times n - r - 1 \dots (r)}$ wherefore this multiplied by $\frac{s}{n \times n - 1 \dots (r)}$, as express'd above, gives $\frac{s}{n \times n - 1 \dots (r)} - \frac{s^2}{n \times n - 1 \dots (2r)}$ for the Probability that the r first taken shall be all a 's, and the r next not all b 's; and this subtracted from $1 - \frac{s}{n \times n - 1 \dots (r)}$, the Probability that the

b 's

b's shall not all be taken, without any Restriction whether the *a's* be, or be not, first taken, leaves $1 - \frac{s^2}{n \times n - 1} (r) + \frac{s^2}{n \times n - 1} (2r)$
 = the Probability that neither the *a's* nor *b's* shall come out according to their Places; whence, by repeating the Operation in the same Manner as in the last Problem, the required Probability will appear to be $\frac{s^2}{n \times n - 1 \times n - 2} (rp)$ in. $1 -$

$$\frac{ms}{n - rp \times n - rp - 1} + \frac{m \times \frac{m-1}{2} ss}{n - rp \times n - rp - 1} (rp + 2r), \quad \&c.$$

Where, if r be taken = 1, s will be = 1, and the Solution the same as the foregoing.

PROBLEM XVIII.

THREE Persons A, B, and C throw in their Turns a Solid having (f) regular Faces, and he who first happens to bring up an assigned Face is thereon to be intitled to a certain Benefit; Required the several Probabilities of obtaining it.

SOLUTION.

LET the Value of the Benefit or Thing expected be denoted by Unity: Then since (*by Cor. to Prob. I.*) the Probability that the assigned Face shall fail $n-1$ Throws successively, and come up the next after, is $\frac{f-1}{f}^{n-1} \times \frac{1}{f}$, this Quantity, it is manifest, will also express the Expectation on the Throw whose Number, from the Beginning, is denoted by n : Therefore in order to find the total Expectation of A, as the 1st, 4th and 7th, &c. Throws pertain to him, let n be expounded by 1, 4, 7, 11, &c. successively; then the above Expression

The NATURE and

sion will become $\frac{1}{f}, \frac{f-1}{f^2}, \frac{f-1}{f^3}, \&c.$ respectively, and the Sum of these will be the first Value sought. In like Manner, if n be expounded by 2, 5, 8, &c. and 3, 6, 9, &c. the total Expectations, or Probabilities, of B and C will come out $\frac{f-1}{f^2} + \frac{f-1}{f^5} + \frac{f-1}{f^8}, \&c.$ and $\frac{f-1}{f^3} + \frac{f-1}{f^6} + \frac{f-1}{f^9}, \&c.$ respectively: But as these 3 Values are infinite Series whose Terms are in geometrical Progression, they may be very easily summed, being equal to $\frac{f}{f^2 - f - 1}, \frac{f \times f - 1}{f^3 - f - 1},$ and $\frac{f - 1}{f^3 - f - 1}$ respectively; which are the Numbers that were required to be found. But the Solution in any Case may be more easily had by considering that the Expectations of $A, B, C, \&c.$ on their first Throws are to one another as their total Expectations; for then by taking as many Terms of the Progression $1, \frac{f-1}{f}, \frac{f-1}{f^2}, \frac{f-1}{f^3}, \&c.$ as there are Players concerned, and dividing each of them by the whole Sum, the several Quotients will be respectively equal to the required Probabilities.

PROBLEM XIX.

TWO Gamesters, A and B, whose Chances for winning any assigned Game are in the given Ratio of (a) to (b), enter into Play on this Condition; That A at the Beginning of every Game shall set the Sum (e) to the Sum (f), and that the Play shall last as long as he continues to win without Intermission; 'Tis required to find the Gain or Advantage of A.

SOLUTION.

SINCE the Expectation of A on any Game, when it comes to

to be play'd is $\frac{a}{a+b} \times e + f$, if (e) his Stake be deducted there-
 from, the Remainder $\frac{af-be}{a+b}$ will consequently be his Gain:
 Therefore the Advantage or Gain on any Game, whose Num-
 ber from the Beginning is denoted by n , to be computed be-
 fore the Play begins, must be compounded of the said Gain
 $\frac{af-be}{a+b}$ and the Probability that the Play will not be ended
 before that Game comes to be decided; that is, of $\frac{af-be}{a+b}$,
 and $\frac{a^{n-1}}{a+b^{n-1}}$ the Probability of his winning all the preced-
 ing $n-1$ Games: Wherefore if n be taken equal to 1, 2, 3, 4,
 &c. successively, we shall have $\frac{af-be}{a+b}$ into 1, $\frac{a}{a+b}$, $\frac{a^2}{a+b^2}$,
 $\frac{a^3}{a+b^3}$, &c. for the Gain on the 1st, 2d, 3d and 4th Games,
 &c. respectively; \therefore that Progression, infinitely continued,
 or $\frac{af-be}{a+b}$ in. $1 + \frac{a}{a+b} + \frac{a^2}{a+b^2} + \frac{a^3}{a+b^3}$, &c. $= \frac{af}{b} - e$, must
 be the true Value. Q. E. I.

PROBLEM XX.

A and B, whose Chances for winning any single Game are
 in Proportion as (a) to (b), the former having (p) and the
 latter (q), Stakes, are determined to play together 'till one of
 them has lost all; To find their respective Probabilities of
 winning, with the Gain or Advantage of A, &c.

SOLUTION.

LET the Expectation of A, when he has any given Num-
 ber y of Stakes in Possession, be express'd by \mathcal{Q} ; and when
 he has one Stake more, or $y+1$, suppose his Expectation to
 be increased by R, or to become $\mathcal{Q}+R$; and when he has yet
 M

yet one more, or $y+2$, let this last Expectation be increased by S , &c. &c. Now when he has $y+1$ Stakes, he must after one Game have either y or $y+2$ Stakes; if he wins that Game he must have $y+2$, and his Expectation will then be $\mathcal{Q}+R+S$; this therefore multiply'd by $\frac{a}{a+b}$, the Probabili-

ty of winning, gives $\frac{\mathcal{Q}a+Ra+Sa}{a+b}$ for his Expectation, in Case of winning, while the Event remains yet undetermined. If he loses the Game, he will have only y Stakes, and his Expectation will then be \mathcal{Q} ; this multiply'd by $\frac{b}{a+b}$, the Pro-

bability of losing, gives $\frac{b\mathcal{Q}}{a+b}$ for the Expectation, in Case of

losing: But it is manifest, that $\frac{\mathcal{Q}a+Ra+S_1+\mathcal{Q}b}{a+b}$, the Sum of those two, must be equal to $\mathcal{Q}+R$, his total Expectation in this Circumstance; whence by Reduction we have $Sa=Rb$, or, R to S , as a to b ; and hence it appears that the Values of the aforesaid Increments, R, S, T, V , &c. are such, that any one of them is to that which immediately succeeds it, in the given Ratio of a to b ; and therefore must constitute a Series of Terms in geometrical Progression, of which the common Ratio is $\frac{b}{a}$. Wherefore, let y be taken $=0$; then

will $R+S+T$ (p), or its equal $R+\frac{Rb}{a}+\frac{Rb^2}{a^2}+\frac{Rb^3}{a^3}$ (p), it is manifest, be the total Expectation of A in the proposed Circumstance; and for the same Reason $R+\frac{Rb}{a}+\frac{Rb^2}{a^2}+\frac{Rb^3}{a^3}(p+q)$ will be his Expectation, when he has $p+q$ Stakes in Possession; but this by the Question is $=p+q$; wherefore R equal

$\frac{p+q}{1+\frac{b}{a}+\frac{b^2}{a^2}+\frac{b^3}{a^3}(p+q)}$; which substituted instead thereof in

the

the other Expression gives $p+q$ in.
$$\frac{1 + \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} (p)}{1 + \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} (p+q)}$$

$= \overline{p+q} \times a^q \times \frac{a^2 - b^2}{a^{p+q} - b^{p+q}}$ for the Expectation of A in the proposed Circumstance; which subtracted from $p+q$ leaves $\overline{p+q} \times b^p \times \frac{a^2 - b^2}{a^{p+q} - b^{p+q}}$ for that of B ; therefore the Gain of A is $-\overline{p+q} \times a^q \times \frac{a^2 - b^2}{a^{p+q} - b^{p+q}}$, and the Ratio of the Probabilities of winning as $a^q \times a^2 - b^2$ to $b^p \times a^2 - b^q$, respectively. Q.E.I.

C O R O L L A R Y I.

WHEN a is $=b$, then $R + \frac{Rb}{a} + \frac{Rb^2}{a^2} + \frac{Rb^3}{a^3} (p)$ will be $=pR$; and the Odds directly as the Stakes to lose.

C O R O L L A R Y II.

IF $p=q$, then will $a^q \times a^2 - b^2$, to $b^p \times a^2 - b^q$, become as a^2 to b^2 , for the Odds in this Case.

C O R O L L A R Y III.

WHEN b is considerably smaller than a , and p and q large Numbers, b^p and b^q will be inconsiderable in respect of a^2 and a^q ; and therefore in that Case the Odds will be as a^2 to b^2 , or as 1 to $\frac{b^2}{a^2}$ nearly.

E X A M P L E.

SUPPOSE B to be throwing with 2 Dice, and every time that

as b^q will bear no Comparifon with a^q , it will be as $a^q \times a^p - b^p$, to $b^p \times a^q$, that is, as $a^p - b^p$, to b^p ; therefore the required Probability is $1 - \frac{b^p}{a^p}$.

PROBLEM XXII.

TO find the Chances that there are for Throwing precifely any Number of Points (p), with any Number of Dice (n), each Die having a given Number (f) of Faces.

SOLUTION.

FIRST, let there be a Set of Dice, having each p , or a greater Number of Faces; then the Chances for p , and all its inferior Numbers, on n fuch Dice, will be equal to the Chances for throwing $p+1$ Points precifely with $n+1$ of the fame Dice; fince it is evident that with all the Chances for p on the n firft Dice, the Ace of the new added Die may be combined, and with all the Chances for $p-1$ the Duce of the fame, &c. And therefore if, in the annexed Scheme, 1, 2, 3, or 4, &c. be put to denote the Number of Dice, and $p-2$, $p-1$, p , or $p+1$, &c. the Number of Points, and the Quantities in the Interfection of the Columns be the Chances for throwing the faid Points with thofe Dice; that is, if for throwing $p-1$ Points with 3 Dice the Chances be C , for p Points with 4 Dice, D , &c. &c. it is manifelt, that B will = $A + A + A + A$, &c. $D = C + C + C$, &c. $D = C + C$, &c. and therefore $D - D = C$, and confequently, for the fame Reafon,

	1	2	3	4	5	6	7
acc.	acc.	acc.	acc.	acc.	acc.	acc.	acc.
$p-2$	A	B	C	D	E	F	G
$p-1$	A	B	C	D	E	F	G
p	A	B	C	D	E	F	G
$p+1$	A	B	C	D	E	F	G

$E-E=D, F-F=E, \&c. \&c.$ Wherefore it appears that the Values of $B, C, \&c.$ are such, that increasing p by Unity they will be augmented by 1, $A, B, \&c.$ Whence by the Method of Increments those Values are easily had equal to $p-1$, $\frac{p-1}{1} \times \frac{p-2}{2}, \frac{p-1}{1} \times \frac{p-2}{2} \times \frac{p-3}{3}, \&c.$ respectively. Therefore, it is manifest, that the Number of Chances for throwing p Points precisely with n such Dice will be $\frac{p-1}{1} \times \frac{p-2}{2} \times \frac{p-3}{3} (n-1)$. But now in order to find the required Value from hence, let the Chances express'd by this Series be called S , and any one assigned Die A , another B , a third C , &c. and suppose the Points on each Face where the Number is greater than f to be red, and the rest black ones; and, for the Sake of Perspicuity, let the red or black Faces of any Die be called the red or black Part of that Die: This being premised, it is evident that the Chances for p Points precisely with n of these Dice, so as to have the red Part of the Die A , always upwards, will be exactly equal to the Chances on the same Dice for $p-f$, without any Restriction. For let the Points on each red Face of that Die be conceived to be diminished by f , then as one or other of the Faces so diminished is, in this Case, always upwards, the Number of Chances for p before such Diminution will consequently be equal to the Chances for $p-f$ after it, that is, if $p-f$ be put $=q$, equal to $\frac{q-1}{1} \times \frac{q-2}{2} \times \frac{q-3}{3} (n-1)$ by what has just now been determined. And from the same way of Reasoning it is plain that the Number of Chances for p Points, so as to have the red Parts both of A and B always upwards, is $\frac{r-1}{1} \times \frac{r-2}{2} \times \frac{r-3}{3} (n-1)$: And for throwing the same Points,

so as to have the red Parts of *A*, *B*, and *C* upwards,
 $\frac{s-1}{1} \times \frac{s-2}{2} \times \frac{s-3}{3} (n-1)$ &c. &c. *r*, *s*, *t*, &c. being equal to
 $p-2f$, $p-3f$, $p-4f$, &c. respectively. Now let these Quan-
 tities, $\frac{q-1}{1} \times \frac{q-2}{2} \times \frac{q-3}{3} (n-1)$, $\frac{r-1}{1} \times \frac{r-2}{2} \times \frac{r-3}{3} (n-1)$, &c.
 be respectively denoted by *G*, *H*, *I*, *K*, &c. And let the
 Chances in *S*, where the red Part of *A* is up alone without other
 red ones, be called *g*; those, where the red Parts of *A* and
B are up together without others of the same Colour, *h*; those,
 where the red Parts of *A*, *B*, and *C* are all up together with-
 out other red ones, *i*, &c. &c. Then, it is evident, that *g* is
 also the Number of Chances for having the red Part of *B*
 alone, or of *C* alone, &c. and therefore all the Chances in *S*
 for having one red Face precisely will be ng : Also as *h* is the
 Chances wherein either *AC*, *AD*, *BC*, or *CD*, &c.
 are red, and the rest black; and because these, the Combina-
 tions of *n* Things taken 2 by 2, are in Number $= n \times \frac{n-1}{2}$,
 the Chances in *S*, with 2 red Faces precisely, in each, will
 therefore be $n \times \frac{n-1}{2} h$; and furthermore, since the Number
 of Combinations of *n* Things taken 3 by 3, 4 by 4, &c. is
 $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, and $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. it follows that
 the Sum of all the Chances in *S*, with one or more red Faces,
 in each, will be $ng + n \times \frac{n-1}{2} h + n \times \frac{n-1}{2} \times \frac{n-2}{3} i$, &c. which
 subtracted from *S*, the whole Number of Chances, leaves
 $S - ng - n \times \frac{n-1}{2} h - n \times \frac{n-1}{2} \times \frac{n-2}{3} i - n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} k$,
 &c. for the Number of Chances with black Points alone
 which is manifestly equal to the Number sought. But now
 to get rid of *g*, *h*, &c. let *Q* be the Chances in *S* that are
 made

made with the red Parts of any Number a of assigned Dice, as A, B, C , &c. without other Parts of the same Colour; or, which is the same, let \mathcal{Q} denote any one of the said Quantities g, b , &c. and let p denote the Chances in \mathcal{S} , having in each of them, also, precisely a red Parts, but only a given Number c of them assigned ones, the other $a-c$ being varied as often as possible; then as the Number of Dice or Parts to be thus varied, or combined, is $n-c$, the Number of such Combinations will be $\frac{n-c}{1} \times \frac{n-c-1}{2} \times \frac{n-c-2}{3} (a-c)$, by *Prob. III.*

And therefore the Number of Chances in the latter of these 2 Cases, just $\frac{n-c}{1} \times \frac{n-c-1}{2} \times \frac{n-c-2}{3} (a-c)$ times as great as

that in the former, or, $P = \mathcal{Q}$ in. $\frac{n-c}{1} \times \frac{n-c-1}{2} \times \frac{n-c-2}{3} (a-c)$. Now taking $c=1$, and a equal to 1, 2, 3, 4, &c. successively, \mathcal{Q} becomes g, b, i, k , &c. and P equal to $g, \frac{n-1}{1} b, \frac{n-1}{1} \times \frac{n-2}{2} i, \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} k$, &c. respectively;

therefore as these Quantities respectively shew the Chances in \mathcal{S} , having the red Part of A alone, or with 1 other, or 2 others, &c. precisely, their Sum must consequently be equal to G , all the Chances in \mathcal{S} with the red Part of A ; and therefore $g = G - \frac{n-1}{1} b - \frac{n-1}{1} \times \frac{n-2}{2} i$, &c. In like Manner by

taking $c=2$, and $a=2, 3, 4$, &c. successively, &c. we have $b = H - \frac{n-2}{1} i - \frac{n-2}{1} \times \frac{n-3}{2} k$, &c. $i = I - \frac{n-3}{1} k - \frac{n-3}{1} \times \frac{n-4}{2} l$, &c. Whence by substituting these Values one by one in

$\mathcal{S} - n g - n \times \frac{n-1}{2} b - n \times \frac{n-1}{2} \times \frac{n-2}{3} i$, &c. as above found, we have $\mathcal{S} - n G + n \times \frac{n-1}{2} H - n \times \frac{n-1}{2} \times \frac{n-2}{3} I + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$

$\frac{n-3}{4} K$, &c. where by resuming the known Values of S , G , H , &c. there will be

$$+\frac{p-1}{1} \times \frac{p-2}{2} \times \frac{p-3}{3} (n-1)$$

$$-\frac{q-1}{1} \times \frac{q-2}{2} \times \frac{q-3}{3} (n-1) \text{ in } n$$

$$+\frac{r-1}{1} \times \frac{r-2}{2} \times \frac{r-3}{3} (n-1) \text{ in } n \times \frac{n-1}{2}$$

$$-\frac{s-1}{1} \times \frac{s-2}{2} \times \frac{s-3}{3} (n-1) \text{ in } n \times \frac{n-1}{2} \times \frac{n-2}{3}$$

&c. where q is $=p-f$, $r=p-2f$, $s=p-3f$, &c.

and the Number of Series to be continued till some one Factor becomes Nothing or Negative. Q. E. I.

C O R O L L A R Y.

HENCE the Chances for not coming up a greater Number of Points than p at any assigned Throw is very readily determined, being equal to

$$p \times \frac{p-1}{2} \times \frac{p-2}{2} (n)$$

$$-\frac{q \times \frac{q-1}{2} \times \frac{q-2}{3} (n) \text{ in } n}{}$$

$$+r \times \frac{r-1}{2} \times \frac{r-2}{3} (n) \text{ in } n \times \frac{n-1}{2}$$

$$\&c. \qquad \&c.$$

E X A M P L E I.

LET it be required to find the Odds that a Person at the first Throw with 2 common Dice does not throw just 8 Points; then f being $=5$, $p=8$, and $n=2$, the general Expressions becomes $=7-2=5$ the Number of Chances for 8 Points,

O

which

which therefore taken from 6^4 , the whole Number of all the Chances, leaves 31 Chances for the contrary; whence the required Odds is as 31 to 5.

Note, ^{Number of} When the Points proposed to be thrown, according to the Problem, is nearer to the greatest Number that can be had, than the least, it will be convenient to use instead thereof that Number, as far distant from the lesser Extreme, as it is from the greater, the Chances for both being manifestly the same.

EXAMPLE II.

How many Chances are there to have 50 Points, precisely, on 10 Dice? Here 20 being as much greater than one Extreme (10) as 50 is exceeded by the other (60), we shall have $p=20$, $n=10$, $f=6$, $g=14$, $r=8$, $s=2$; and therefore $\frac{10}{1} \times \frac{18}{2} \times \frac{17}{3} \times \frac{16}{4} \times \frac{15}{5} \times \frac{14}{6} \times \frac{13}{7} \times \frac{12}{8} \times \frac{11}{9} - \frac{17}{1} \times \frac{15}{2} \times \frac{14}{3} \times \frac{13}{4} \times \frac{12}{5} \times \frac{11}{6} \times \frac{10}{7} \times \frac{9}{8} \times \frac{8}{9} = 85228$, the Number that was to be found.

EXAMPLE III.

WHAT is the Odds that at the first Throw with 10 Dice there shall come up more than 15 Points? Here taking $n=10$, $f=6$, $p=15$, and substituting these Values in the second general Expression, we shall have 3003 for the Number of Chances by which 15 Points and all its inferior Numbers may happen; this taken from $60466176=6!^{10}$ the whole Number of Chances leaves 60463173; therefore as 60463173 to 3003, so are the Odds.

A TABLE exhibiting the Chances by which any Number of Points may be precisely had with 10 common Dice.

Points.	Chances.	Points.	Chances.	Points.	Chances.	Points.	Chances.
10 . 60	1	17 . 53	11340	24 . 46	576565	31 . 39	3393610
11 . 59	10	18 . 52	23760	25 . 45	831204	32 . 38	3801535
12 . 58	55	19 . 51	46420	26 . 44	1151370	33 . 37	4121260
13 . 57	220	20 . 50	85228	27 . 43	1535040	34 . 36	4325310
14 . 56	715	21 . 49	147940	28 . 42	1972630	35 . 35	4395456
15 . 55	2002	22 . 48	243925	29 . 41	2446300	Total of all the Chances being 6 ¹⁰ = 60466176.	
16 . 54	4995	23 . 47	383470	30 . 40	2930455		

PROBLEM XXIII.

THERE is a Solid having (*m*) similar and equal Faces, whereof (*p*) are marked *A*; (*q*) of them, *B*; *r*, *C*, &c. What is the Probability that in throwing up a given Number (*n*) of such Solids there shall arise a given Number (*b*) of assigned Sorts of Faces, as *A*, *B*, *C*, &c.

SOLUTION.

THE Probability that no *A* shall come up being $\frac{m-p}{m}^n$, (by Prob. I.) that of the contrary must be $\frac{m-m-p}{m}^n$; and therefore as the whole Number of Chances is m^n , that for having one or more *A* upwards will consequently be $m^n - m-p$; that is, there are $m^n - m-p$ different Ways by which the Faces of the Solids may be varied to have one *A* or more up at each Variation. Therefore if all the Faces marked *B* be now restrained from coming up, then there being only *m-q* Faces that can arise, the last Expression will (by substituting *m-q* instead of *m*), it is evident, become $m-q-p$, equal to all the Variations that can possibly be made to have

A

A up, when B is restrained from appearing; which therefore being subtracted from $m^n - \overline{m-p}^n$, the ~~chances for~~ ~~chances for~~ having A up, without any Restriction upon B , leaves

$m^n - \left| \frac{\overline{m-p}}{\overline{m-q}} \right|^n + \overline{m-p-q}^n$ for the Number of Chances to have both A and B upwards; since it is self evident that in all the said Variations for A , either B must, or must not, be upwards. In like Manner, if the C 's be all restrained from coming upwards, m , the Number of Faces in each Solid, may conceived to be reduced to $m-r$, and then the last Expression will become $\overline{m-r}^n - \left| \frac{\overline{m-p-r}}{\overline{m-q-r}} \right|^n + \overline{m-p-q-r}^n$ for the Number of Variations that can be made with A and B up, when C is restrained from appearing; Wherefore this taken from $m^n - \left| \frac{\overline{m-p}}{\overline{m-q}} \right|^n + \overline{m-p-q}^n$, as found above, must leave

$m^n - \left| \frac{\overline{m-p}}{\overline{m-q}} \right|^n + \left| \frac{\overline{m-p-q}}{\overline{m-p-r}} \right|^n - \overline{m-p-q-r}^n$, for the Number of different Ways by which A , B and C may be all upwards. By the same Method of Reasoning the Chances for having A , B , C and D all upwards will be found

$$m^n - \left| \frac{\overline{m-p}}{\overline{m-q}} \right|^n + \left| \frac{\overline{m-p-q}}{\overline{m-p-r}} \right|^n - \left| \frac{\overline{m-p-q-r}}{\overline{m-p-q-s}} \right|^n + \overline{m-p-q-r-s}^n$$

Whence the Law of Continuation is manifest. Wherefore dividing the Quantity thus resulting by m^n , the whole Number of Chances, the Quotient, it is manifest, will be the Probability required.

COROLLARY. I.

WHEREFORE, when p, q, r , &c. are equal to one another, the Number of Chances in the last Case will, it is evident, become $m^n - 4 \times \overline{m-p}^n + 6 \times \overline{m-2p}^n - 4 \times \overline{m-3p}^n + \overline{m-4p}^n$; where the Unciæ are those of a Binomial raised to the Power whose Index is equal to the Number of Sorts A, B, C, D , whose Chances for happening, all, upwards are exhibited by that Expression; and the like appears in any other Case. Therefore when p, q, r , &c. are as above specified,

$$\frac{m^n - b \times \overline{m-p}^n + b \times \frac{b-1}{2} \times \overline{m-2p}^n - b \times \frac{b-1}{2} \times \frac{b-2}{3} \times \overline{m-3p}^n}{m^n}$$

&c. must consequently be the Value proposed to be found.

COROLLARY II.

BECAUSE when m is a large, and p a small Number, the Quantities $m^n, \overline{m-p}^n, \overline{m-2p}^n$, &c. are nearly in a geometrical Progression, or equal to $m^n, \frac{\overline{m-p}^n}{m^n}, \frac{\overline{m-2p}^n}{m^n}$, &c. respectively, the above Expression, in this Case, will be nearly

$$m^n - b \times \overline{m-p}^n + b \times \frac{b-1}{2} \times \frac{\overline{m-2p}^n}{m^n}, \text{ \&c.}$$

equal to $\frac{m^n - b \times \overline{m-p}^n + b \times \frac{b-1}{2} \times \frac{\overline{m-2p}^n}{m^n}}{m^n} = 1 - b \times$

$$\frac{\overline{m-p}^n}{m^n} + b \times \frac{b-1}{2} \times \frac{\overline{m-2p}^n}{m^n} - b \times \frac{b-1}{2} \times \frac{b-2}{3} \times \frac{\overline{m-3p}^n}{m^n}, \text{ \&c. or, that}$$

Power of the Binomial $1 - \frac{\overline{m-p}}{m}$ whose Index is b , that is $= \overline{1 - \frac{\overline{m-p}}{m}}^b$; and therefore if the required Probability be denoted by \mathcal{P} , in this Case \mathcal{P} will $= \overline{1 - \frac{\overline{m-p}}{m}}^b$ very nearly :

$$\text{Whence } \mathcal{P}^{\frac{1}{b}} = 1 - \frac{\overline{m-p}}{m}, \quad \overline{1 - \frac{\overline{m-p}}{m}}^b = 1 - \mathcal{P}^{\frac{1}{b}}, \text{ and } n = \frac{\text{Log. } 1 - \mathcal{P}^{\frac{1}{b}}}{\text{Log. } 1 - \frac{\overline{m-p}}{m}}$$

$= -\frac{1}{n}$ in hyp. Log. $1 - P^{\frac{1}{n}}$ nearly; by which, if P be given, n may be obtained.

E X A M P L E I.

ONE with 2 common Dice undertakes to throw both the Numbers 5 and 7 at 3 Tryals; What are the Odds against him? Here if we suppose a Die, or Solid, having 36 Faces, whereof 4 are marked A , and 6 B ; it is manifest that the Probability of throwing both A and B in 3 Throws with that Solid, is the same as that of throwing both the Numbers 5 and 7 at 3 Tryals with 2 common Dice: Wherefore, according to the general Theorem we have $m=36$, $p=4$, $q=6$, $n=3$, and

$$\frac{m^n - \frac{m-p}{m-q}^n + m-p-q}{m^n} = \frac{18^3 - 16^3 - 15^3 + 13^3}{18^3} = \frac{31}{324}; \text{ whence the}$$

Odds is as 293 to 31.

E X A M P L E II.

LET it be required to find the Probability that a Person in throwing 6 common Dice shall bring up the Ace, Duce and Tray.

In this Case, n is $=6$, $b=3$, $m=6$, $p=1$, $q=1$, &c. Therefore (by Cor. I.) $\frac{6^6 - 3 \times 5^6 + 3 \times 4^6 - 3^6}{6^6} = \frac{n}{144}$ will be the Value sought.

E X A M P L E III.

To find in how many Throws with a single Die one may undertake to throw all the 6 Faces.

By comparing this Case with Cor. II. we have $n=1$, $m=6$, $b=6$, $P=\frac{1}{6}$, and (n) the required Number equal

$$\text{Log. } \frac{1 - \frac{1}{6}}{1 - \frac{1}{6}} = 12.$$

PROB.

PROBLEM XXIV.

TO find the Probability that a proposed Event shall happen a given Number of Times (p) without Intermission in a given Number (n) of Tryals.

SOLUTION.

LET r be the Probability of Happening of the proposed Event at any assigned Tryal, and m that of the contrary; and upon the Happening of the said Event p Times successively, let a Person, B , receive a certain Sum S : Then will the Probability of receiving that Sum at the End of an assigned Number of Tryals be compounded of 3 others; as, first, The Probability of Happening of the proposed Event p Times in so many Tryals: Second, That of its Failing the Time immediately preceding those Tryals: And, lastly, That of his not having received it before that Time. For if the first succeed not, the Thing is manifestly impossible; if the second prove contrary, he must either not receive it at all, or at the End of some of the preceding Tryals; and, lastly, when he has once received it, there can be no farther Probability of obtaining it from any future Tryal.

Therefore, as the Probability of the proposed Event's failing any one assigned Tryal, and then happening p Times without Intermission, is $m \times r^p$, (by Prob. I.), if r^p be put $=a$, and $m \times r^p = x$, we shall have x equal to the Probability of his receiving the said Sum at the End of $p+1, p+2, p+3$ or $p+p$ Tryals, because, it is manifest, the last of the aforementioned Probabilities does not take Place till after the $2p$ first Tryals: But the Manner how each Value is derived from the preceding ones, and the Relation they bear

bear to one another, will appear better by help of the following Scheme; wherein the second Column towards the left Hand shews the Probability of receiving that Sum at the End of a given Number of Tryals represented by the first; and the third, the Probability that he receives it some time in those Tryals. Each Line, or Expression, of the second Column being formed by drawing x into the Excess of Unity above the Value of that Line of the third Column, whose Distance, or Place above the Line so formed, is denoted by $p+1$; according to the Reasons above specified, this last Column being generated by a continual Addition of the Terms of the former.

Tryals.	Probability.	Probability.
p	a	a
$p+1$	x	$a+x$
$p+2$	x	$a+2x$
&c.	x	&c.
$2p$	x	$a+px$
$2p+1$	$x \times 1 - a$	$a - ax + \overline{p+1} \times x$
$2p+2$	$x \times 1 - a - x$	$a - 2ax + \overline{p+2} \times x - x^2$
$2p+3$	$x \times 1 - a - 2x$	$a - 3ax + \overline{p+3} \times x - 3x^2$
$2p+4$	$x \times 1 - a - 3x$	$a - 4ax + \overline{p+4} \times x - 6x^2$
$3p$	$x \times 1 - a - \overline{p-1} \times x$	$a - pax + \overline{2p} \times x - p \times \frac{p-1}{2} x^2$

Now from having proceeded thus far, the Law of Continuation is manifest; The Value of the 3d Col. or the required Probability, in any Case (where n is not less than p) being a in

$$1 - \frac{n}{n}x + \frac{n}{n} \times \frac{n-1}{2}x^2 - \frac{n}{n} \times \frac{n-1}{2} \times \frac{n-2}{3}x^3 + \frac{n}{n} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}x^4, \&c. \text{ Plus } \frac{n}{n} \times \frac{n-1}{2}x^2 + \frac{n}{n} \times \frac{n-2}{3}x^3 - \frac{n}{n} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}x^4, \&c. \text{ where } n \text{ is put } = n-p, n=n-2p, n=n-3p, \&c.$$

Q. E. I.

EXAMPLE I.

LET it be required to throw a proposed Chance 3 times without Intermiffion in 10 Tryals; when the Odds for its happening any affigned Tryal is as 2 to 1: Then taking $r = \frac{2}{2+1}$, $m = \frac{1}{2+1}$, $p = 3$, $n = 10$; we have $n = 7$, $n = 4$, $a = \frac{8}{17}$, $x = \frac{8}{17}$, and therefore $\frac{8}{17} \times 1 - \frac{12}{17} + \frac{16}{17} - \frac{128}{27 \times 81} = \frac{592}{729}$ for the Probability in this Cafe.

EXAMPLE II.

IN 200 Throws with a fingle Die, what is the Odds that the Ace does not come up 8 Times fucceffively? Here r being $= \frac{1}{6}$, $m = \frac{1}{6}$, $p = 8$, $n = 200$, our Series become $\frac{1}{6^8} \times 1 - \frac{5 \times 184}{6^9}$, &c. $+ \frac{5 \times 192}{6^9} - \frac{92 \times 183 \times 25}{6^{18}}$, &c. $= .0000966442$ nearly; which fubtracted from 1, leaves .9999033558; therefore the required Odds will be as .999903, &c. to .0000966442, or as 10356 to 1 nearly.

Note. Tho' to have the Answer accurately true, both the Series ought to be continued till they terminate, or fome of the Factors become Negative; yet if a near Approximation be only wanted a few of the firft Terms may

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fuffice,

suffice, as in the last Example, where by taking only 2 Terms of each Series, the true Solution is exhibited to less than $\frac{1}{10000}$ Part of the Whole.

PROBLEM XXV.

A and B, whose Proportion of Skill, or Chance for winning any assigned Game, is as (a) to (b), agree to play together, till the former is a Winner of (p) Stakes, or the latter of (q) Stakes; To find the Probability, that the Play between them ends in a given Number (n) of Games.

SOLUTION.

CASE I. Let $p=2$ and $q=2$; Then the required Value will be $1 - \frac{2ab}{a+b} \frac{2ab}{a+b} \frac{n}{2}$ when n is an even Number, and $1 - \frac{2ab}{a+b} \frac{n-1}{2}$ when an Odd one.

CASE II. Let $p=3$ and $q=3$; Then $1 - \frac{3ab}{a+b} \frac{3ab}{a+b} \frac{n-1}{2}$, or $1 - \frac{3ab}{a+b} \frac{n-2}{2}$, according as n is odd or even, will be the Value, in this Case.

CASE III. Supposing a and b, or the Chances of the Gamesters to be equal. From the Binomial $1+1$ raised to the n Power, cut off as many Terms as there are Units in $\frac{n-p+\frac{1}{2}}{2}$, and of these take the q last, reject the p preceding, take the q next, reject the p next, &c. till all the Terms are exhausted; then the Sum of all the Terms thus taken, divided by 2^{n-1} , will be the Probability of the Play ending in

in favour of A , in the proposed Number of Games, if $n-p$ be an odd Number. And if the Terms, of those cut off, whose Distances from the last are denoted by $0, q, q+p, 2q+p, 2q+2p, 3q+2p, \&c.$ be taken with Signs $-$ and $+$ alternately, and added to twice the above said Sum, and the whole be divided by 2^n , you will have the Probability of the same in the other Case when $n-p$ is not an odd Number.

Generally. Put $\frac{n-p+\frac{1}{2}+\frac{1}{2}}{2} = m$, and let the m first Terms of $\overline{a+b}^n$ expressed in a Series be denoted by R ; the $m-q$ first, by S ; the $m-q-p$ first, by T ; the $m-2q-p$ first, by U , &c. Also let $l = \frac{n-p+\frac{1}{2}+\frac{1}{2}}{2}$, and the l last Terms of the said Series be denoted by r ; the $l-q$ last, by s ; the $l-q-p$ last, by t ; the $l-2q-p$ last, by v , &c. Then the Probability of the Play ending in favour of A , in the given Number of Games, will be $R - S \times \frac{\overline{b}^q}{a} + T \times \frac{\overline{b}^{q+p}}{a} - U \times \frac{\overline{b}^{2q+p}}{a} + W \times \frac{\overline{b}^{2q+2p}}{a}, \&c. + r \times \frac{\overline{a}^p}{b} - s \times \frac{\overline{a}^{p+q}}{b} + t \times \frac{\overline{a}^{2p+q}}{b} - v \times \frac{\overline{a}^{2p+2q}}{b} + w \times \frac{\overline{a}^{3p+2q}}{b}, \&c.$ when the whole is divided by $\overline{a+b}^n$. And it is manifest, that what is said in either of the 2 last Cases, will hold equally, in respect to the Play ending in favour of B , if p be changed for q , q for p , b for a , and a for b .

COROLLARY.

If q , the Number of Stakes which A has to lose, be greater than n the Number of Games, then will $S=0, T=0, \&c.$ and the general Expression $= R + \frac{r \overline{a}^p}{b}$; which shews the Probability that he has of being a Winner of p Stakes in the given Number

Number of Games, when the Duration of the Play is not restrained by what he may happen to lose: And this, when $n-p$ is an odd Number, and $a=b$, will, it is manifest, be just double to the Probability that there is of his coming off an actual Winner of q , or more, Stakes at the end of n Games; supposing here, contrary to the Proposition, the Duration of the Play to be limited to that Number.

Note 1. *That when any Number of Terms, to be taken as in Case the third, is greater than that of the Terms remaining, these last must be used instead of the former.*

Note 2. *That the Signs + and - both prefixed to $\frac{1}{2}$ in any Expression, are to be used, according as that, or this shall be found necessary to make that Expression a whole Number.*

EXAMPLE I.

LET $a=1$, $b=1$, $n=11$, and the Number of Stakes each Gamester has to lose $=3$; then, by Case the 2d, $1 - \frac{3^{11-1}}{2^{11-1}} = \frac{781}{1024}$ will be the Probability that the Play is ended in 11 Games: But to find the same, according to Case the 3d, from $1 + n + nx \frac{n-1}{2}$, &c. $(= \overline{1+1})^n = 1 + 11 + 55 + 165$, &c. I cut off the 5 $(= \frac{n-p + \frac{1}{2} + \frac{1}{2}}{2})$ first Terms, take the Sum of the 3 (q) last of them, as $55 + 165 + 330 = 550$, and reject the rest; and because 8 $(= n-p)$ is not an odd Number, to twice this Sum $= 1100$, I add $-330 + 11$, and divide the Aggregate by 2", and there comes out $\frac{781}{1024}$ for $\frac{1}{2}$ the Value fought.

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EXAMPLE II.

LET $a=b$, $p=6$, $q=10$, and $n=21$. From $1+21+21 \times 10$, &c. cutting off the first 8 Terms $1+21+210+1330+5985+20349+54264+116280$, taking their Sum and dividing the same by 21^n we have $\frac{198440}{1048576} = \frac{24805}{131072}$ for the Probability of the Play ending to the Advantage of A in 21 Games. And by taking the Sum of the first 6 of those Terms and dividing as before, there comes out $\frac{27896}{1048576} = \frac{3487}{131072}$ for the Probability of the same Thing happening in Favour of B ; but the Sum of those two is $\frac{7073}{32768}$, and therefore the Odds that the Play is not ended in 21 Games as $32768-7073$ to 7073 , or nearly as 11 to 5.

EXAMPLE III.

SUPPOSE $p=3$, $q=4$, $n=11$, $a=2$, and $b=1$; then, according to the 4th Case, R will be $=a^{11}+11a^{10}b+55a^9b^2+165a^8b^3+330a^7b^4$, or $2^7 \times 16+88+220+330+330=128 \times 984$, $S=a^{11}=128 \times 16$, $T=0$, &c. $r=b^{11}+11b^{10}a+55b^9a^2+165b^8a^3=1+22+220+1320=1563$, $s=0$, &c. and there-

fore $\frac{R-Sx\frac{b^9}{a} \&c. +rx\frac{a^9}{b} \&c.}{a+b^{11}} = \frac{138328}{177147}$ is the Probability that the Play ends to the Advantage of A in 11 Games; and if to this be added $\frac{6771}{177147}$, shewing the like in respect to B , there will arise $\frac{145099}{177147}$ for the required Probability of the Play being ended in 11 Games.

P R O B L E M XXVI.

A and B, whose Chances for winning any single Game are in Proportion as (a) to (b), enter into Play together. What is the Probability that A shall first be a Loser of (q) Stakes, before he is a Winner of (p) Stakes, and afterwards a Winner of (p) Stakes, and all this in the first (n) Games?

S O L U T I O N.

It is found in the last Problem, that the Probability which A has of being a Winner of p Stakes in the proposed Number of Games, retaining the same Construction, is

$$R - S \times \frac{\bar{b}^q}{a} + T \times \frac{\bar{b}^{q+p}}{a} \&c. r \times \frac{\bar{a}^p}{b} - s \times \frac{\bar{a}^{p+q}}{b} \&c.$$

$\frac{a + b^n}{a + b^n}$; when the Play is supposed to terminate upon his losing q Stakes, if this should happen before the n Games are expired. And it appears by the Corollary to the same, that the Probability he has of being a Winner of those Stakes, in the same Number of Games, when the Duration of the Play is not restrained by what he may happen to lose, is $R + \frac{ra}{b}$: Therefore by so much as this last Value exceeds the former, by so much, it is manifest, will the Probability be express'd, that he shall be first a Loser of q Stakes, and afterwards a Winner of p such,

$$S \times \frac{\bar{b}^q}{a} - T \times \frac{\bar{b}^{q+p}}{a} + U \times \frac{\bar{b}^{q+p+q}}{a} \&c. + s \times$$

or this Excess, which is $\frac{a + b^n}{a + b^n}$

$$\frac{\bar{b}^{p+q}}{a} - t \times \frac{\bar{b}^{p+q}}{a} + v \times \frac{\bar{a}^{p+q}}{b} \&c.$$

$$\frac{a + b^n}{a + b^n} \text{ will be the Value. Q. E. I.}$$

COROLLARY.

WHEN a is $=b$, the Answer will be more easily had by the following Contraction of the foregoing Method, *i. e.* From $1 + 1$ raised to the n Power, cut off as many Terms as there are Units in $\frac{n-p+1}{2} - q$, and of these take the p last, reject the q preceding ones, take the p next, reject the q next, &c. Then, if $n-p$ be an odd Number, the Sum of all the Terms thus taken, divided by 2^{n-1} , will give the Value sought. And, when $n-p$ is an even Number, if the Terms of those so cut off, whose Distances from the last are denoted by $0, p, p+q, 2p+q$, &c. be taken with Signs $-$ and $+$ alternately, and added to twice the said Sum, and the whole be divided by 2^n , the Quotient will be the Value sought in this Case.

EXAMPLE.

LET $a=b, p=5, q=3$, and $n=24$. From $1+n+nx$ $\frac{n-1}{2}$ &c. ($=1+1^n$) cutting off the first 7 Terms, and taking the Sum of the 5 last of them, we have 190026, which divided by 4194304 ($=2^{23}$) gives the required Value.

PROBLEM XXVII.

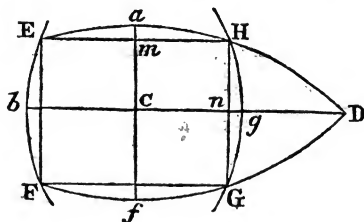
IN a Parallelopipedon, whose Sides are to one another in the Ratio of a, b, c ; To find at how many Throws any one may undertake that any given Plane, viz. ab , may arise.

SOLUTION.

IMAGINE the given Parallelopipedon to be circumscribed by a spherical Surface, and to be sustained by a Power acting
at

at its Center of Gravity in the Direction of a right Line standing perpendicular to the Horizon; and while the Center of Gravity and that Line are supposed to remain at Rest, let the Solid, together with the circumscribing Surface, be so moved, that the said perpendicular Line may trace out the Perimeter, or Limits, of the proposed Plane; Then it will not be difficult to conceive that that Part of the spherical Surface, which will be limited by the Intersection, or Path described, by the said Perpendicular, thro' such Motion, will be to the whole Surface, as the Probability of the given Plane arising at any assigned Throw to Unity: It is also obvious that the Limits of the said Part will be Arcs of 4 great Circles, whose Chords are the Sides of the given Plane. This being premised; let the Parallelopipedon be now supposed at Rest, with its upper or given Plane, $E H G F$, parallel to the Horizon; and let the great Circles

EaH , HgG , GfF , and FbE be the Limits of the abovesaid Part of the spherical Surface, whose Content we would find; let also 2 other great Circles,



bg , fa , bisect the opposite Arches EF , HG ; and EH , FG , and EH and FG be produced to meet bg some where as in D : Then the Angles C , b , a , g , and f being all right ones, the Sides Da , DC , DF will be each 90 Degrees, and Ca the Measure of the Angle aDC . Wherefore, if (720) twice the Degrees in the Circle be put to define the Content of the whole

whole ſpherical Surface, Ca or its equal HDg will expreſs the Content of the Triangle aDC , and $DHg + HDg - 90^\circ$, that of the Triangle HDg (*as is proved in p. 179. of my Book of Fluxions.*) Whence, by taking the laſt of theſe 2 from the former, we have $90^\circ - gHD$ for the Content of the Part $CaHgC$; wherefore as 720° to 4 times $90^\circ - gHD$, or as 180° to the Complement of the Angle gHD ; ſo is the Content of the whole Surface, to that of the required Part $EHGFE$. Now therefore to determine the ſaid Complement; we have by Plane Trigonometry, As $\frac{1}{2}c$, the perpendicular Diſtance of the given Plane from the Center of Gravity of the propoſed Solid, to $Cm (= \frac{1}{2}b)$ half one Side of that Plane, ſo is (1) the Radius of the Tables, to $\frac{1}{2}a$, the Tangent of Ca , or HDg ; and as the ſaid Half-Diſtance is to $Cn (\frac{1}{2}a)$ half the other Side, ſo is Radius to $\frac{1}{2}c$, the Tangent of Cg , or Co-tang. of Dg : Therefore in the Right-Angled Triangle HDg there is given ($\frac{1}{2}a$) the Tangent of the Angle D , and ($\frac{1}{2}c$) the Co-tang. of the Side Dg , from whence the Sine of the former and Co-fiue of the latter will be eaſily had $\frac{b}{cc+bb^{\frac{1}{2}}}$

and $\frac{a}{aa+cc^{\frac{1}{2}}}$ reſpectively, and then, *per Spherics*, it will be as
 $1 : \frac{b}{cc+bb^{\frac{1}{2}}} :: \frac{a}{aa+cc^{\frac{1}{2}}} : \frac{ab}{aa+cc \times bb+cc^{\frac{1}{2}}} =$ the Sine of the Complement propoſed to be found.

Now therefore let A be put for the Angle whoſe Sine is $\frac{ab}{aa+cc \times bb+cc^{\frac{1}{2}}}$, and x for the required Number of Throws; then, it follows from what has been ſaid above, that the Probability of the given Plane ariſing at any aſſigned Throw will be $\frac{A}{180}$; therefore that of the contrary being $1 - \frac{A}{180}$, we

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shall have $1 - \frac{A}{180}x$ equal $\frac{1}{2}$ and consequently x equal $\frac{-L:2}{L:1 - \frac{A}{180}}$,
or $\frac{144}{A} \times 1 - \frac{A}{180}$ nearly. Q. E. I.

COROLLARY. I.

WHEN a , b , and c are equal, the Parallelopipedon becomes a Cube, and $\frac{ab}{aa+cc+bb+cc} = \frac{1}{3} =$ the Sine of 30 Degrees; therefore in that Case $A=30$, and $x = \frac{144}{30} \times 1 - \frac{30}{180} = 3.8$.

COROLLARY II.

IF the given Plane (ab) be but small in respect of the others, then the Sine $\frac{ab}{aa+cc+bb+cc}$ being also small, will be nearly as A its Corresponding Angle, that is, $\frac{ab}{aa+cc+bb+cc}$ will be to 3.141, &c. as A to 180°; whence A equal $\frac{180ab}{3.1416(aa+cc+bb+cc)}$ nearly; which Value being substituted in $\frac{144}{A}$ (then nearly $=x$) gives $\frac{2.2 \times aa+cc+bb+cc}{ab}$ for the required Number of Throws in this Case.

LEMMA I.

Supposing A, B, C , &c. to be the Coefficients of the Power of $a+x$, whose Exponent is n ; or, $A+Bx+Cx^2+Dx^3$, &c. $= (a+x)^n$: To find the Value of $\frac{A}{1.2.3(r)} + \frac{Bx}{2.3.4(r)} + \frac{Cx^2}{3.4.5(r)} + \frac{Dx^3}{4.5.6(r)} + \frac{Ex^4}{5.6.7(r)}$, &c. in finite Terms, r being any whole

whole positive Number, and n any positive Number, or negative one, except an Integer less than $r+1$.

SINCE $A+Bx+Cx^2+Dx^3$, &c. is $=\overline{a+x}^n$, $A+Bx+Cx^2$, &c. $\times x^r$ will therefore $=\overline{a+x}^n \times x^r$; whence, by taking the Fluent continually of the last Equation, we have,

$$\text{First, } \frac{\overline{a+x}^{n+1}}{n+1} \times x^{r-1} - \frac{a^{n+1}}{n+1} \times x^{r-1} = x^{r-1} \text{ in. } \frac{Ax}{1} + \frac{Bx^2}{2} + \frac{Cx^3}{3},$$

$$\text{\&c. Second, } \frac{\overline{a+x}^{n+2}}{n+1 \times n+2} \times x^{r-2} - \frac{a^{n+2}}{n+1 \times n+2} \times x^{r-2} = \frac{a^{n+1}}{n+1} \times x^{r-2}$$

$$= x^{r-2} \text{ in. } \frac{Ax^2}{1.2} + \frac{Bx^3}{2.3} + \frac{Cx^4}{3.4}, \text{\&c. And, lastly, } \frac{\overline{a+x}^{n+r}}{n+1 \times n+2(r)}$$

$$= \frac{a^{n+r}}{n+1 \times n+2(r)} - \frac{a^{n+r-1} x}{n+1 \times n+2(r-1)} - \frac{\frac{1}{2} a^{n+r-2} x^2}{n+1 \times n+2(r-2)}$$

$$= \frac{\frac{1}{6} a^{n+r-3} x^3}{n+1 \times n+2(r-3)} - \frac{\frac{1}{24} a^{n+r-4} x^4}{n+1 \times n+2(r-4)} (r+1) = \frac{Ax^r}{1.2.3(r)} +$$

$$\frac{Bx^{r+1}}{2.3.4(r)} + \frac{Cx^{r+2}}{3.4.5(r)}, \text{\&c. Wherefore, dividing this Equati-}$$

$$\text{on by } x^r, \text{ we have } \frac{\overline{a+x}^{n+r-r}}{n+1 \times n+2(r)} - \frac{a^{n+r-1} x}{n+1 \times n+2(r)}$$

$$= \frac{a^{n+r-1} x^{-r+1}}{n+1 \times n+2(r-1)} - \frac{\frac{1}{2} a^{n+r-2} x^{-r+2}}{n+1 \times n+2(r-2)} (r+1) = \frac{A}{1.2.3(r)} +$$

$$\frac{Bx}{2.3.4(r)} + \frac{Cx^2}{3.4.5(r)}, \text{\&c. Q. E. I.}$$

LEMMA II.

TO find the Sum of a Series of Powers whose Roots are in arithmetical Progression, as $a^n + 2a^n + 3a^n + 4a^n + 5a^n$ &c. continued to x Terms.

LET d^n into $Ax^{n+1} + Bx^n + Cx^{n-1} + Dx^{n-2} + Ex^{n-3}$, &c. $+K$

$+K = d^n + 2d^n + 3d^n + 4d^n \dots \dots \dots x d^n$; then must d^n into
 $\overline{Ax+1}^{n+1} + \overline{Bx+1}^n + \overline{Cx+1}^{n-1}$, &c. $+K$, it is manifest,
 be $= d^n + 2d^n + 3d^n + 4d^n \dots \dots x d^n + \overline{x+1} \times d^n$; wherefore,
 by taking the former of these Equations from the latter, and
 dividing the whole by d^n , we have $\overline{Ax+1}^{n+1} - x^{n+1} + B \times$
 $\overline{x+1}^n - x^n + C \times \overline{x+1}^{n-1} - x^{n-1} + D \times \overline{x+1}^{n-2} - x^{n-2}$, &c. =
 $\overline{x+1}^n$; where expanding the several Powers of $x+1$ in
 Series, and comparing the like Terms, we get $A = \frac{1}{n+1}$, $B =$
 $\frac{1}{2}$, $C = \frac{n}{3.4}$, $D = 0$, $E = \frac{n \times n - 1 \times n - 2}{2.3.4.5.6}$, F equal 0, G equal
 $\frac{n \times n - 1 \times n - 2 \times n - 3 \times n - 4}{2.3.4.5.6.7.6}$, &c. And therefore $d^n + 2d^n +$
 $3d^n \dots \dots \dots x d^n = \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{n x^{n-1}}{3.4} - \frac{n \times n - 1 \times n - 2 \times x^{n-3}}{2.3.4.5.6}$
 $+ \frac{n \times n - 1 \times n - 2 \times n - 3 \times n - 4 \times x^{n-5}}{2.3.4.5.6.7.6}$, &c. $+K$, into d^n ; which
 when $x=1$, will become $d^n = \frac{1}{n+1} + \frac{1}{2} + \frac{1}{3.4} - \frac{n \times n - 1 \times n - 2}{2.3.4.5.6}$
 &c. $+K$ in d^n : Wherefore $K = \frac{-1}{n+1} + \frac{1}{2} - \frac{n}{3.4} + \frac{n \times n - 1 \times n - 2}{2.3.4.5.6}$
 &c. Consequently d^n in $\frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{n x^{n-1}}{3.4} - \frac{n \times n - 1 \times n - 2 \times x^{n-3}}{2.3.4.5.6}$
 $+ \frac{n \times n - 1 \times n - 2 \times n - 3 \times n - 4 \times x^{n-5}}{2.3.4.5.6.7.6} -$
 $\frac{n \times n - 1 \times n - 2 \times n - 3 \times n - 4 \times n - 5 \times n - 6 \times x^{n-7}}{2.3.4.5.6.7.8.5.6} +$
 $\frac{n \times n - 1 \times n - 2 \times n - 3 \times n - 4 \times n - 5 \times n - 6 \times n - 7 \times n - 8 \times x^{n-9}}{2.3.4.5.6.7.8.9.11.12}$, &c.

$$-\frac{1}{n+1} + \frac{1}{2} - \frac{n}{3 \cdot 4} + \frac{n \times n - 1 \times n - 2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \frac{n \times n - 1 \times n - 2 \times n - 3 \times n - 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} +$$
 &c. equal to the Value sought; which, when n is a whole positive Number, will be barely expressed by d^n in $\frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{nx^{n-1}}{3 \cdot 4} - \frac{n \times n - 1 \times n - 2 \times n}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} x^{n-3}$, &c. continued till it terminates, provided that the last Term thereof when n is an odd Number be rejected. Q. E. I.

COROLLARY I.

HENCE by taking $d=1$, and $n=1, 2, 3, 4$, &c. successively, we shall have

$$\begin{aligned}
 1+2+3+4+5+6 \dots x &= \frac{x^2}{2} + \frac{x}{2} \\
 1^2+2^2+3^2+4^2+5^2 \dots x^2 &= \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6} \\
 1^3+2^3+3^3+4^3+5^3 \dots x^3 &= \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4} \\
 1^4+2^4+3^4+4^4+5^4 \dots x^4 &= \frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30} \\
 1^5+2^5+3^5+4^5+5^5 \dots x^5 &= \frac{x^6}{6} + \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12} \\
 &\text{\&c.} \qquad \qquad \qquad \text{\&c.}
 \end{aligned}$$

COROLLARY II.

IF x be infinite, and n any positive Number, or negative one less than Unity; the first Term of the Series, it is manifest, will be infinite in respect of the rest of the Terms; and therefore the Equation will in that Case become $1^n + 2^n + 3^n$

$$+ 4^n \dots x^n = \frac{x^{n+1}}{n+1}.$$

T

L A M M A

TO find an Expression or Series, which consisting of the simple Powers of x and known Coefficients, shall be equal to $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$, &c. continued to x Factors.

Let P denote the given Value, and $\overline{x-a} L: x, +Ax+B + Cx^{-1} + Dx^{-2}$, &c. the hyp. Log. thereof then will $\overline{x+1-ax} L: x+1, +A \times \overline{x+1} + B + C \times \overline{x+1}^{-1} + D \times \overline{x+1}^{-2} + E \times \overline{x+1}^{-3}$, &c. be $= L: P + L: x+1$; from which Equation by substituting the Value of $L: p$, &c. there will be $\overline{x-a} L: \overline{1+1}x + A + C \times \overline{x+1}^{-1} - x^{-1} + D \times \overline{x+1}^{-2} - x^{-2} + E \times \overline{x+1}^{-3} - x^{-3}$, &c. $= 0$; this converted into simple Terms

$$\text{is } \left| \begin{array}{cccccccc} 1 - \frac{x^{-1}}{2} + \frac{x^{-2}}{3} - \frac{x^{-3}}{4} + \frac{x^{-4}}{5} - \frac{x^{-5}}{6} + \frac{x^{-6}}{7}, & \&c. \\ * - ax^{-1} + \frac{ax^{-2}}{2} - \frac{ax^{-3}}{3} + \frac{ax^{-4}}{4} - \frac{ax^{-5}}{5} + \frac{ax^{-6}}{6}, & \&c. \\ A - * - Cx^{-2} + Cx^{-3} - Cx^{-4} + Cx^{-5} - Cx^{-6}, & \&c. \\ * * * - 2Dx^{-3} + 3Dx^{-4} - 4Dx^{-5} + 5Dx^{-6}, & \&c. \\ * * * * - 3Ex^{-4} + 6Ex^{-5} - 10Ex^{-6}, & \&c. \\ * * * * * - 4Fx^{-5} + 10Fx^{-6}, & \&c. \\ * * * * * - 5Gx^{-6}, & \&c. \\ * * * * * & \&c. \end{array} \right| = 0$$

Whence $A=1$, $a=-\frac{1}{2}$, $C=\frac{1}{3.4}$, $D=0$, $E=\frac{-1}{3.4.5.6}$, $F=0$,
 $G=\frac{1}{5.6.7.8}$, $H=0$, $I=\frac{-1}{5.6.7.8}$, &c. and consequently $\overline{x+\frac{1}{2}} \times$
 $L: x, -x+B+\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\frac{1}{11880x^9}$, &c.
 $= L: P$. Suppose $x=1$, and the Equation will become -1
 $+B+\frac{1}{12}-\frac{1}{360}$, &c. $= 0$, and therefore B must be $= 1 -$
 $\frac{1}{12} + \frac{1}{360} - \frac{1}{1260}$ &c. which substituted instead thereof, gives

$\overline{x+\frac{1}{2}}$

$\overline{x+1} \times L : x, -x+1 - \frac{1}{12} + \frac{1}{360}, \&c. + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}$
 $\&c. = L : P.$ Now the Number whose hyp. Log. is 1 will
 be = 2.71828, &c. that whose Log. is $\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5},$
 $\&c. = 1 + \frac{1}{12x} + \frac{1}{288x^3} - \frac{139}{51840x^5}, \&c.$ and that answering to $1 -$
 $\frac{1}{12} + \frac{1}{360}, \&c.$ equal to 2.506, &c. which, as Mr *Demoivre*
 observes from Mr *Sterling*, is the square Root of the Peri-
 phery of a Circle, whose Semi-Diameter is Unity: Where-
 fore if that Periphery be denoted by c , and 2.71828, &c.
 the Number whose hyp. Log. is 1, by m , and our Logarith-
 mical Equation be reduced to one shewing the Relation of
 the corresponding Numbers, by multiplying together those
 above found answering to the several Parts thereof, we shall
 have x^{x+1} in m^{-x} in $c^{\frac{1}{2}}$ in $1 + \frac{1}{12x} + \frac{1}{288x^3} - \frac{139}{51840x^5}, \&c.$
 $= P. Q. E. I.$

C O R O L L A R Y.

BECAUSE when x is a very great Number $\frac{1}{12x} + \frac{1}{288x^3},$
 $\&c.$ becomes inconsiderable, P or its Equal $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$
 $\dots \times x$ will then be nearly equal to $\frac{x^x}{m}$ in $\sqrt{cx}.$

L E M M A IV.

IF c be the Periphery of a Circle, whose Radius is Unity,
 the Ratio of the middle or greatest Term of the Binomial
 $1+1$, raised to a very great Power (n), will be to the whole
 Power, as 1 to $\frac{\sqrt{cn}}{2}$ very nearly.

FOR

For it is known, from the Principles of Quadratures, and is proved in page 118 and 121, of my Book of *Fluxions*, that c being, as above specified, the Area of the whole Curve, whose Abscissa is x , and Ordinate $\frac{x^n}{\sqrt{1-xx}}$, when n is an even Number, will be $\frac{c}{4}$ into $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \dots \times \frac{n-1}{n}$, and that the Area of the Curve, having the same Abscissa, whose Ordinate is $\frac{x^{n+1}}{\sqrt{1-xx}}$, under the same Values of x and n , is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \dots \times \frac{n}{n+1}$. Therefore since these two Areas, when n is very great, are very near equal to each other, $\frac{c}{4} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \dots \times \frac{n-1}{n}$ will be nearly $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \dots \times \frac{n}{n+1}$; and therefore, by Division, $\frac{c}{4} = \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \dots \times n \times n}{1 \times 1 \times 3 \times 3 \times 5 \times 5 \dots \times n-1 \times n-1 \times n+1}$ where multiplying both Sides by $n+1$, we get $\frac{c}{4} \times n+1$, or $\frac{cn}{4} = \frac{2 \times 2 \times 4 \times 4 \dots \times n \times n}{1 \times 1 \times 3 \times 3 \dots \times n-1 \times n-1}$ nearly; the square Root whereof is $\frac{\sqrt{cn}}{2} = \frac{4 \times 4 \times 6 \dots n}{1 \times 3 \times 5 \dots n-1} = \frac{2 \times 2 \times 4 \times 4 \dots n \times n}{1 \times 2 \times 3 \times 4 \dots n-1 \times n} = 2^n \times \frac{1^2 \times 2^2 \times 3^2 \dots \times \frac{1}{2}n^2}{1 \times 2 \times 3 \times 4 \dots n} = 2^n \times \frac{1 \times 2 \times 3 \times 4 \dots \frac{1}{2}n}{n \times n-1 \times n-2 \times n-3 \dots \frac{1}{2}n+1}$; wherefore it is as $1 : \frac{\sqrt{cn}}{2} :: \frac{n \times n-1 \times n-2 \times n-3 \dots \frac{1}{2}n+1}{1 \times 2 \times 3 \times 4 \dots \frac{1}{2}n} : 2^n$ but the third Term of this Proportion is known to be equal to the middle Term of the Binomial proposed, and the fourth thereof equal to the whole Power. Q. E. I.

LEMMA V.

TO find the Ratio that an assigned Term of a Binomial, raised to an infinite Power, bears to the whole Series.

LET

LET $a+b$ be the given Binomial, n the Index of its Power, r the Distance of the proposed Term from the first, and put $s = n - r$: Then since the Value of the said Term is $\frac{n \times n - 1 \times n - 2 \dots n - r + 1 \times a^r b^s}{1.2.3.4.5 \dots r}$, we shall, by equally multi-

plying by $1 \times 2 \times 3 \times 4 \times 5 \dots s$, have $\frac{n \times n - 1 \dots 3 \times 2 \times 1 \times a^r b^s}{1.2.3 \dots r \times 1.2.3 \dots s}$ which by *Lemma III.* is $= \frac{\sqrt[n]{n} \times \sqrt[n]{cn}^{\frac{1}{2}} \times a^r b^s}{\sqrt[n]{r} \times \sqrt[n]{cr}^{\frac{1}{2}} \times \sqrt[n]{s} \times \sqrt[n]{cs}^{\frac{1}{2}}} = \frac{n^{n+r} \times a^r b^s}{r^r \times s^s \times \sqrt{crs}}$; m and c being as there specified; wherefore dividing by $a+b$ we have $\frac{n^{n+r} \times a^r b^s}{r^r \times s^s \times a + b^s \times \sqrt{crs}}$. Q. E. I.

COROLLARY.

BECAUSE the greatest Term of the whole Power is that wherein the Exponents of a and b are to one another as a to b , if s be taken to r in the Ratio of a to b , or $r = \frac{bn}{a+b}$, and $s = \frac{an}{a+b}$, and these Values be substituted above, we shall have $\frac{a+b}{\sqrt{abcn}}$, for the Ratio of the greatest Term to the whole Power; which therefore, when a and b are equal, will become $\frac{2}{\sqrt{cn}}$ the very same as in *Lemma IV.* Hence it is manifest, that the Value of c in *Lemma III.* is not only near, but exactly equal to the Periphery of the Circle, whose Radius is Unity; which there was not easy to determine.

LEMMA VI.

TO find the Proportion which the greatest Term of a Binomial, raised to an infinite, or very great Power, bears to a given Number of Terms next it.

U

LET

LET $a+b$ be the proposed Binomial, as above, n the Index of its Power, y the greatest Term, and p the Number of Terms to be taken on either Side. Because the greatest Term is that wherein the Indices of a and b are equal to $\frac{na}{a+b}$, and $\frac{nb}{a+b}$; if $\frac{na}{n+b}$ be put $=s$, and $\frac{nb}{a+b}=r$, the next Term to it will, it is manifest, be $=\frac{s \times b y}{r+1 \times a}$; the next after that $=\frac{s \times s - 1 \times b y}{r+1 \times r+2 \times a^2}$, the next following equal $\frac{s \times s - 1 \times s - 2 \times b y}{r+1 \times r+2 \times r+3 \times a^3}$, and consequently that, whose Distance from the greatest is p , equal to $\frac{s \times s - 1 \times s - 2 \dots s - p + 1}{r+1 \times r+2 \times r+3 \dots r+p}$ in $\frac{b y}{a^p}$; and therefore its hyperbolical Logarithm equal to

$$L:y + \left| \begin{array}{l} pL:b \\ pL:s \\ -pL:a \\ -pL:r \end{array} \right| + \left| \begin{array}{l} \frac{1}{s} - \frac{1}{2s^2} + \frac{1}{3s^3} - \frac{1}{4s^4}, \&c. \\ -\frac{2}{s} + \frac{4}{2s^2} - \frac{8}{3s^3} + \frac{16}{4s^4}, \&c. \\ -\frac{3}{s} + \frac{9}{2s^2} - \frac{27}{3s^3} + \frac{81}{4s^4}, \&c. \\ -\&c. \text{ to } p-1 \text{ Series} \dots \\ -\frac{1}{r} + \frac{1}{2r^2} - \frac{1}{3r^3} + \frac{1}{4r^4}, \&c. \\ -\frac{2}{r} + \frac{4}{2r^2} - \frac{8}{2r^3} + \frac{16}{4r^4}, \&c. \\ -\frac{3}{r} + \frac{9}{2r^2} - \frac{27}{3r^3} + \frac{81}{4r^4}, \&c. \\ -\&c. \text{ to } p \text{ Series} \dots \end{array} \right|$$

But because ar is $=bs$, $pL:b, +pL:s, -pL:a, -pL:r$ vanishes out of the Equation. And the Numerators of the remaining Terms being Series of Powers, whose Roots are in arithmeti-

cal Progression, their Sum will be easily had by *Lemma II.*

and from thence our Expression becomes $L:y, \frac{p-1^2-p+1}{2s}$

$$-\frac{2 \times p-1^2-3 \times p-1^2-p+1}{12s^2} \frac{p-1^4+2 \times p-1^2+p-1^2}{12s^3}, \&c.$$

$$-\frac{p^2+p}{2r} - \frac{2p^3+3p^2+p}{12r^2} - \frac{p^4+2p^3+p^2}{12r^3}, \text{ which, } p \text{ being small}$$

in respect to r and s , will become $=L:y, -\frac{pp}{2r} - \frac{pp}{2s}$ very near-

ly; where by substituting for r and s their Equals $\frac{na}{a+b}$, and

$\frac{nb}{a+b}$, we shall have $L:y, -\frac{pp}{n} \times \frac{a+b}{2ab} = (L:T)$ the Log. of that

Term whose Distance from the greatest is denoted by p ; and

therefore $T=y \times 1 - \frac{dp^2}{n} + \frac{d^2p^4}{2n^2} - \frac{d^3p^6}{2.3n^3} + \frac{d^4p^8}{2.3.4n^4}$, &c. d , for the

sake of Brevity, being put instead of $\frac{a+b}{2ab}$: Hence the Sum

of all the Terms between the two expressed by y and T , with

the last of these inclusive, will, by *Lemma II.* be $=yp$ in. $1 -$

$$\frac{dp^2}{3n} + \frac{d^2p^4}{2.5n^2} - \frac{d^3p^6}{2.3.7n^3} + \frac{d^4p^8}{2.3.4.9n^4} - \frac{d^5p^{10}}{2.3.4.5.11n^5} + \frac{d^6p^{12}}{2.3.4.5.6.13n^6}$$

&c. very nearly. Where, if v be put $=\frac{p}{n}$, or $p=\sqrt{vn}$,

it will become $y\sqrt{vn}$ in. $1 - \frac{dv}{3} + \frac{d^2v^2}{2.5} - \frac{d^3v^3}{2.3.7} + \frac{d^4v^4}{2.3.4.9}$, &c.

from whence the required Proportion is manifest.

C O R O L L A R Y.

Because the greatest Term divided by the whole Series, or

$$\frac{y}{a+b}, \text{ is } = \frac{a+b}{\sqrt{abcn}} \text{ (by Cor. to Lem. 4.) } y \text{ will be } =$$

$$a+b$$

$\frac{a+b_{n+1}}{\sqrt{abcn}}$; which being substituted instead thereof in the last of the above Expressions, we shall have $\frac{a+b_{n+1}}{\sqrt{abc}} \times \sqrt{\frac{v}{abc}}$ in. $1 - \frac{dv}{3} + \frac{d^2v^2}{2.5} - \frac{d^3v^3}{2.3.7} + \frac{d^4v^4}{2.3.4.9}$, &c. equal to the Sum of as many Terms, immediately succeeding the greatest, as there are Units in \sqrt{vm} ; c being =, 2×3.1415 , &c. and $d = \frac{a+b}{2ab}$. And therefore when a and b are equal the said Sum, it is manifest, will become $\frac{a+b}{2} \times 2 \sqrt{\frac{v}{c}}$ in. $1 - \frac{2v}{3} + \frac{4v^2}{2.5} - \frac{8v^3}{2.3.7} + \frac{16v^4}{2.3.4.9} - \frac{32v^5}{2.3.4.5.11} + \frac{64v^6}{2.3.4.5.6.13}$, &c.

PROBLEM XXVIII.

Supposing a given Number, $n+1$, of Adventurers, playing at Raffles, or any other Game of the like Nature, and the first of them, A, to have raffled and got so great a Number, that there are only (a) Chances for a greater, (b) Chances for the same, and (c) for a lesser Number; To find the Probability of his Winning: And, supposing the whole Stake or Thing raffled for, to be in Proportion to the Number of Players; To find also what that Number must be to make his Advantage by this Circumstance the greatest possible.

SOLUTION.

SINCE $\frac{b+c}{a+b+c}$ is the Probability that no assigned Player gets a greater Number than A, the Probability that no one of the (n) Players shall bring up a greater Number, will be $c+b$

$$\frac{\overline{c+b}^n}{a+b+c}^n; \text{ or, if } \overline{c+b}^n \text{ be converted into simple Terms,}$$

$$c^n + nc^{n-1}b + n \times \frac{n-1}{2} c^{n-2}b^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} c^{n-3}b^3, \&c. \text{ (by Cor.}$$

$$\frac{\overline{a+b+c}^n}{a+b+c}^n$$

Prob. 1.); the Terms of this Series being respectively equal to the Chances, for 0, 1, 2, 3, &c. of those Players having an equal Number to that of *A*, and all the rest, at the same Time, lesser Numbers: Wherefore if these Terms c^n , $nc^{n-1}b$, $n \times \frac{n-1}{2} c^{n-2}b^2$, &c. be respectively divided by 1, 2, 3, &c. the

Sum of the several Quotients apply'd to $\overline{a+b+c}^n$ will be

$$\frac{c^n}{1} + \frac{nc^{n-1}b}{2} + \frac{n \times n-1 \times c^{n-2}b^2}{2.3} + \frac{n \times n-1}{2} \times \frac{n-2}{3} \times \frac{c^{n-3}b^3}{4}, \&c. \text{ for the}$$

$$\frac{\overline{a+b+c}^n}{a+b+c}^n$$

first Part of the Answer. But this, by *Lemma 1.* is =

$$\frac{\overline{c+b}^{n+1} - c^{n+1}}{n+1 \times b} \text{ in } \frac{1}{\overline{a+b+c}^n}; \text{ or, if } a+b+c \text{ be put } = s, \text{ and}$$

$$c+b=d, \text{ equal } \frac{d^{n+1} - c^{n+1}}{n+1 \times s^nb}; \text{ this therefore multiplied by } n+1,$$

the whole Stake is $\frac{d^{n+1} - c^{n+1}}{bs^n}$, the Expectation of *A*, in the

second Case; which by the Question must be a Maximum,

and therefore its Fluxion $\frac{nd^{n+1}}{bs^n} \times L \cdot \frac{d}{s} - \frac{nc^{n+1}}{bs^n} \times L \cdot \frac{c}{s} = 0$, or

$d^{n+1} \times L \cdot \frac{d}{s} = c^{n+1} \times L \cdot \frac{c}{s}$; Wherefore putting the Hyperbolic Lo-

garithm of $\frac{s}{d} = b$, and that of $\frac{s}{c} = g$, it will be $\frac{\overline{d}^{n+1}}{c} = \frac{b}{g}$; whence

$$n+1 = \frac{\text{Log. } b - \text{Log. } g}{\text{Log. } d - \text{Log. } c}. \quad \text{Q. E. I.}$$

EXAMPLE.

LET A 's Number be such, that there may be only one Chance for a greater, one for the same, and 100 for a lesser Number; then a being = 1, b = 1, c = 100, d = 101, and s = 102, g will be 0.0198, and h = 0.0985, and therefore $n+1$, according to the last Case, = 70; whence it appears that the greatest Expectation that A can possibly have in the above Circumstance, is, when there are 69 Players besides himself; it then being about 25.7 Times his own Stake. But if the Players were 100, his Expectation, by Case 1, would be only 24 Times his Stake, and if they were infinite it would be indefinitely small. And from hence appears the great Disadvantage that even a good Gamester, or one that has a great Number of Chances for Winning, will have in playing where there are others better than himself, even tho' the greater Part of the Players are at the same Time much worse than himself.

PROBLEM XXIX.

A and B, whose Chances for winning any assigned Game are in the given Proportion of a , to b , agree to play till n stakes are won and lost, on Condition that A , at the Beginning, of every Game shall set the Sum p to the Sum $p \times \frac{b}{a}$, so that they may play without Disadvantage on either Side; 'Tis required to find the present Value of all the Winnings that may be betwixt them when the Play is ended.

SOLUTION

LET s be that Term of $\overline{a+b}^n$ expressed in a Series, where the Exponent of the Power of a , divided by that of b , is, as near

near as may be, equal to, but not less than $\frac{a}{b}$; and let the Distance of that Term from the first, or the Index of b , in the said Term, be denoted by d : Then because a^n , $na^{n-1}b$, $n \times \frac{n-1}{2} a^{n-2}b^2$, $n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3}b^3 \dots$, the former Terms of that Series are (as has been found in *Prob. 5*) the respective Chances for A 's winning $n, n-1, n-2, n-3$, &c. Games precisely, or gaining the Sums $\frac{pb}{a} \times n$, $\frac{pb}{a} \times n-1-p$, $\frac{pb}{a} \times n-2-2p$, $\frac{pb}{a} \times n-3-3p$; if those Terms be respectively drawn into these Sums, *viz.* a^n in $\frac{pb \times n}{a}$, $na^{n-1}b$ in $\frac{pb \times n-1-p}{a}$, &c. the several Products be added together, and the whole be divided by, $\overline{a+b}^n$, the total Number of Chances, we shall have $\frac{pbs \times n-d}{a \times a+b}$ for half the Value sought; which when the Ratio of the above said Exponents, is exactly, as a to b , will become $\frac{pbns}{a+b|^{n+1}}$; for then $\overline{n-d} \times b = da$, and $\frac{n-d}{a} = \frac{n}{a+b}$ Q. E. I.

COROLLARY II.

HENCE, if n be a very great Number, $\frac{s}{a+b|}$ being $\frac{1}{2}$ $\frac{a+b}{\sqrt{abcn}}$, by *Lemma 5*. the Value in this Case will be $2p \sqrt{\frac{bn}{ac}}$; c being, as there specified, equal to the Periphery of a Circle whose Semi-diameter is Unity.

EXAMPLE I.

LET $a=2$, $b=1$, $p=1$, and $n=8$; then taking the third Term of $a^8+8a^7b+28a^6b^2+56a^5b^3$, &c. because the Ratio $\frac{5}{3}$ of

of the Exponents in the next Term is less than $\frac{a}{b}$, there will

be $\frac{28a^6b^6 \times 6}{a+b} = 1, \frac{1397}{2187}$ for the true Value in this Case.

EXAMPLE II.

SUPPOSE $a=1$, $b=1$, $p=1$, and $n=10000$; and we shall have $2p\sqrt{\frac{bn}{ac}} = 2\sqrt{\frac{n}{c}} = 79.8$; but if n had been supposed 1000000, the Answer would have been only 798.

PROBLEM XXX.

TWO Gamesters, A and B, equally skilful, enter into Play together, and agree to continue the same till (n) Games are won and lost. 'Tis required to find the Probability that neither comes off a Winner of $1\sqrt{n}$ Stakes, and also the Probability that B is never a Winner of that Number of Stakes during the whole Time of the Play; 1 being a given, and n any very great, Number.

SOLUTION.

IF from $1+1$ raised to the n Power be cut off the first half of the Terms, and as many of the last of them be taken as there are Units in $\frac{r\sqrt{n}}{2}$ and divided by $1+1^n$, the Quotient will, by

Prob. 5, be one half of the Value sought: But the said $\frac{r\sqrt{n}}{2}$ Terms according to Cor. to Lem. 6, by substituting $\frac{r\sqrt{n}}{2}$

instead of \sqrt{vn} , will be $1+1^n \times \frac{r}{\sqrt{c}}$ in. $1 - \frac{r^2}{3.2} + \frac{r^4}{2.5.4} - \frac{r^6}{2.3.7.8} + \frac{r^8}{2.3.4.9.16} - \frac{r^{10}}{2.3.4.5.11.32} + \frac{r^{12}}{2.3.4.5.6.13.64}$, &c. And

therefore

therefore the Double hereof, divided by $1+1$ gives $\frac{r^2}{\sqrt{c}}$ in. 1

$$-\frac{r^2}{3.2} + \frac{r^4}{2.5.4} - \frac{r^6}{2.3.7.8}$$
 or .798, &c. $\times r$ in. $1 - \frac{r^2}{3.2} + \frac{r^4}{2.5.4} - \frac{r^6}{2.3.7.8} + \frac{r^8}{2.3.4.9.16}$ for the first Part of the Answer: But this is also the Answer in the other Case, as is manifest from Cor. to Prob. 25. Q. E. I.

COROLLARY.

If the Probability should be given $= \frac{1}{2}$, and the Value of r be required; then it will be .798, &c. $\times r$ in. $1 - \frac{r^2}{3.2} + \frac{r^4}{2.5.4} - \frac{r^6}{2.3.7.8}$, &c. $= \frac{1}{2}$; where r will be $= 0.674$, &c. Hence it appears that, however great n may be, it is an equal Chance that neither of the Gamesters comes off a Winner of $0.674 \sqrt{n}$ Stakes: And from this and the last Problem it may be also observed, that tho' a Gamester may at some Times be a very great Winner, or, on the other Hand, a very great Loser, yet, at Long-run, there is almost any assigned Odds, if he always plays upon an Equality of Chance, that there will scarcely be any Comparison between his Loss or Gain, in the End, and the whole of the Money he ventures; And second, that if an Event after a great many Tryals, is continually found to fail or succeed more frequently than it should according to the known Chances, by which it seems to be decided, there is the greatest Reason to suspect that that Event is affected or govern'd by some other Cause which we are unacquainted with.



